128. On Certain Reduction Theorems for Systems of Differential Equations which Contain a Turning Point

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1. **Introductions.** In this paper we consider a system of linear ordinary differential equations

(1.1) $$\varepsilon dx/dt = A(t, \varepsilon)x,$$

where x is an n-vector: $A(t, \varepsilon)$ is a matrix of type (n, n), which admits a uniformly asymptotic expansion

(1.2) $$A(t, \varepsilon) = \sum_{j=0}^{\infty} A_j(t)\varepsilon^j$$

for $|t| < t_0$, as ε tends to zero through a domain $|\arg \varepsilon - \theta| < \varepsilon_0$. The coefficients of this expansion, $A_j(t)$ are holomorphic functions of t in the domain $|t| < t_0$.

The system has a turning point at the origin, if $A_0(t)$ has a set of eigenvalues: $\lambda_{j_1}(t), \cdots, \lambda_{j_p}(t) (p \leq n)$, which are zero for $t=0$, but at least a pair of eigenvalues are not identically equal, where, by a theorem due to Sibuya, (cf. Sibuya, Y. [3]), we may assume $p=n$.

Though a general method to treat such a system is not yet known, all the known results are obtained by reducing the coefficient matrix $A(t, \varepsilon)$ to a matrix, whose elements are polynomials in the independent variable. Moreover, if there is a formal transformation

(1.3) $$y = P(t, \varepsilon)x \quad P(t, \varepsilon) \sim \sum_{j=0}^{\infty} P_j(t)\varepsilon^j$$

such that

(1.4) $$\det P_0(0) \neq 0, \quad P_j(t)\text{: holomorphic for } |t| < t_0$$

which reduces the system (1.1) to a system with polynomial coefficients, then, in a sectorial domain, there is a matrix $Q(t, \varepsilon)$ which has the same asymptotic expansion as $P(t, \varepsilon)$. (cf. Sibuya, Y. [4]). We shall call a formal transformation (1.3) with the properties (1.4), a *formal admissible transformation.*

Our results are stated in two theorems:

Theorem 1. *If in* (1.2) $A_0(t)$ *is in the form*

(1.5.1) $$A_0(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ t & 0 & 0 & \cdots & 0 \end{pmatrix},$$

then there is a formal admissible series (1.3) *such that*

$$\varepsilon dy/dt = A_0(t)y.$$

Theorem 2. *If in* (1.2) $A_0(t)$ *is in the form*

$$A_0(t)=\begin{pmatrix} 0\,1\,0\cdots 0 \\ 0\,0\,1\cdots 0 \\ \cdots\cdot \\ 0\,0\,0\cdots 1 \\ 0\,t\,0\cdots 0 \end{pmatrix}, \tag{1.5.2}$$

then there is a formal admissible transformation (1.3) *such that*

$$\varepsilon dy/dt=B(t,\varepsilon)y,$$

where

$$B(t,\varepsilon)=A_0(t)+\varepsilon\sigma\begin{pmatrix} 0\,0\cdots 0 \\ 0\,0\cdots 0 \\ \cdots\cdot \\ 1\,0\cdots 0 \end{pmatrix}.$$

σ *is a formal power series of* ε *with constant coefficients*

$$\sigma\sim\sum_{k=0}^{\infty}\sigma_k\varepsilon^k. \tag{1.6}$$

Remark. Those theorems can be applied when, in (1.5.1) and (1.5.2), t is replaced by an holomorphic function $\varphi(t)$ such that

$$\varphi(0)=0, \text{ and } \varphi'(0)\neq 0.$$

The system (1.1) with $A_0(t)$ in the form (1.5.1), contains the second order equation treated in Langer, R. E. [1], and with $A_0(t)$ in the form (1.5.2), contains the third order equation treated in Langer, R. E. [2] in its special cases.

2. Angorithms. We shall use the following lemma.

Lemma. *Let* $\mu(t,\varepsilon)$ *be a row-vector with* n *components, using the matrix* $A(t,\varepsilon)$ *in* (1.2), *and define a set of row vectors* $p_k(t,\varepsilon)$, $k=1,2,\cdots,n+1$, *as follows*

$$\begin{cases} p_1(t,\varepsilon)=\mu(t,\varepsilon) \\ p_k(t,\varepsilon)=\varepsilon dp_{k-1}/dt+p_{k-1}(t,\varepsilon)A(t,\varepsilon) \quad (2\leq k), \end{cases} \tag{2.1}$$

then

$$p_{k+1}=\mu A_0^k+\varepsilon(k\mu' A_0^{k-1}+\mu\Psi_k(A_0))+\varepsilon^2 f_k(t,\mu,\varepsilon) \tag{2.2}$$

where

$$\Psi_k(A_0)=\sum_{j=1}^{k-1}((A_0^j)'A_0^{k-j-1}+A_0^jA_1A_0^{k-j-1}) \tag{2.3}$$

and $f_k(t,\mu,\varepsilon)$ *is a linear form in* $\mu,\mu',\mu'',\cdots,\mu^{(k)}$: *the coefficients of this linear form are polynomials in* $A,A',\cdots,A^{(k)}$.

Proof. By iteration

$$p_2=\varepsilon\mu'+\mu A$$
$$p_3=\varepsilon^2\mu''+\varepsilon(2\mu' A+\mu A')+\mu A^2$$
$$\cdots\cdots$$

$$p_{k+1}=\mu A^k+\varepsilon(k\mu' A^{k-1}+\mu\sum_{j=1}^{k-1}(A^j)'A^{k-j-1})+\varepsilon^2(\cdots)+\cdots+\varepsilon^k\mu^{(k)}. \tag{2.4}$$

Substituting the expression (1.2) into (2.4) we have the lemma.

We shall consider a transformation $y=Px$ which carries the system (1.1) to a system

$$\varepsilon dy/dt = B(t, \varepsilon)y, \tag{2.5}$$

where P is a matrix whose row-vectors are $p_1, p_2, \cdots, p_n$; namely

$$P(t, \varepsilon) = \begin{pmatrix} p_1(t, \varepsilon) \\ p_2(t, \varepsilon) \\ \cdots\cdot \\ \cdots\cdot \\ p_n(t, \varepsilon) \end{pmatrix}. \tag{2.6}$$

By the expression

$$\varepsilon dy/dt = (\varepsilon P' + PA)P^{-1}y = \begin{pmatrix} p_2(t, \varepsilon) \\ p_3(t, \varepsilon) \\ \cdots\cdot \\ p_{n+1}(t, \varepsilon) \end{pmatrix} P^{-1}(t, \varepsilon)y$$

$B(t, \varepsilon)$, is necessarily of the form

$$B(t, \varepsilon) = \begin{pmatrix} 0\ 1\ 0 \cdots 0 \\ 0\ 0\ 1 \cdots 0 \\ 0\ 0\ 0 \cdots 0 \\ \cdots\cdot \\ b_1\ b_2 \cdots b_n \end{pmatrix}. \tag{2.7}$$

So in order to find a matrix $P(t, \varepsilon)$, such that (2.5) holds for arbitrarily given set $(b_1, b_2, \cdots, b_n)$, it is sufficient to find a vector μ, such that

$$p_{n+1} = \sum_{k=1}^{n} b_k p_k \tag{2.8}$$

3. Proof of Theorem 1. Set $b_1 = t, b_k = 0$ for $k = 2, 3, \cdots, n$ in (2.8), and using the lemma,

$$\mu A_0^n + \varepsilon(n\mu' A_0 + \mu\psi_n(A_0)) + \varepsilon^2 f_n(t, \mu, \varepsilon) = t\mu. \tag{3.1}$$

Subtracting $\mu A_0^n = t\mu$ (Cayley-Hamilton theorem) and multiplying A_0 from the right,

$$nt\mu' + \mu\psi_n(A_0)A_0 + \varepsilon f_n(t, \mu, \varepsilon)A_0 = 0. \tag{3.2}$$

Substituting the formal expression

$$\mu(t, \varepsilon) \sim \sum_{j=1}^{\infty} \mu_j(t)\varepsilon^j \tag{3.3}$$

we have an infinite sequence of systems of differential equations

$$t\mu_0' = \mu_0 G(t) \tag{3.4.0}$$

$$t\mu_k' = \mu_k G(t) + g_k(t, \mu) \quad k = 0, 1, 2, \cdots \tag{3.4.k}$$

where $G(t) = -1/n \cdot \psi_n(A_0)A_0$, and $g_k(t, \mu)$ is a linear form in $\mu_0, \mu_1, \cdots, \mu_{k-1}$ and their derivatives.

Each system has a regular singular point at the origin, and the eigenvalues of $G(0)$ are calculated from the equality

$$A_0^k = \begin{pmatrix} 0 & E_{n-k} \\ tE_k & 0 \end{pmatrix}$$

where E_{n-k} is a unit matrix of order $n-k$. If we set $A_1(t) = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$, where Q, R, S and T are matrices of type $(k, k), (k, n-k), (n-k, k)$ and $(n-k, n-k)$ respectively:

$$A_0^k A_1 A_0^{n-k} = \begin{pmatrix} 0 & E_{n-k} \\ tE_k & 0 \end{pmatrix} \begin{pmatrix} Q & R \\ S & T \end{pmatrix} \begin{pmatrix} 0 & E_k \\ tE_{n-k} & 0 \end{pmatrix} = \begin{pmatrix} tT & S \\ t^2R & tQ \end{pmatrix}.$$

We see that this matrix has diagonal elements all zero. On the other hand

$$(A_0^k)'A_0^{n-k}=\begin{pmatrix}0 & 0\\ E_k & 0\end{pmatrix}\begin{pmatrix}0 & E_k\\ tE_{n-k} & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & E_k\end{pmatrix}$$

so that $G(0)=-1/n\cdot\sum_{k=1}^{n-1}(A_0^kA_1A_0^{n-k}+(A_0^k)'A_0^{n-k})_{t=0}$ is a triangular matrix with diagonal elements,

(3.5) $$0,\ -1/,\ n,\cdots,\ -2/n,\cdots,\ -(n-1)/n$$

and these are the eigenvalues of $G(0)$.

Since there is 0 among the eigenvalues, and $(1,0,\cdots,0)$ is a corresponding eigenvector of $G(0)$, there is an holomorphic solution with the initial condition $(1,0,\cdots,0)$ of the system (3. 4. 0), moreover this choice of the initial condition enables us to satisfy

$$\det P_0(0)\neq 0.$$

If we suppose that we have found $\mu_0(t), \mu_1(t),\cdots,\mu_{k-1}(t)$, which are holomorphic at the origin, then we can find a solution of (3.4.k) which is holomorphic at the origin, for $g_k(t,\mu)$ is a row vector with components holomorphic functions at the origin, and the first component of $g_k(t,\mu)A_0$ is zero at $t=0$. This completes the proof of Theorem 1.

4. Proof of Theorem 2. We put $b_1=\varepsilon\sigma, b_2=t$ in (2.8) with an undetermined formal series (1.6) and using Lemma

$$\mu tA_0+\varepsilon(n\mu'A_0^{n-1}+\mu\psi_n(A_0))+\varepsilon^2 f_n(t,\mu,\varepsilon)=\varepsilon\sigma\mu+\varepsilon t\mu'+\mu tA.$$

Subtracting μtA_0 from both sides,

(4.1) $$\mu'(nA_0^{n-1}-t)=-\mu(\psi_n(A_0)-\sigma-tA_1)+\varepsilon(-f_n+\mu t\sum_2^\infty A_k\varepsilon^k).$$

The calculation of $\psi_n(A_0)$ follows from the expression

$$A_0^k=\begin{pmatrix}0 & E_{n-k}\\ 0 & 0\end{pmatrix}+\left(\begin{array}{c|c|c} \begin{matrix}0\\0\\0\\\cdot\cdot\end{matrix} & 0 & 0\\ \hline \begin{matrix}0\\\cdot\\\cdot\\0\end{matrix} & tE_k & 0 \end{array}\right)\begin{matrix}\}\,n-k\\ \cdot\\ \}\,k\end{matrix}$$

$$\qquad\qquad\qquad 1 \qquad\ \ \underbrace{\ \ }_{k}\ \underbrace{\ \ }_{(n-k-1)}$$

We have

$$(A_0^k)'A_0^{n-k-1}=\begin{pmatrix}0 & 0\\ 0 & E_k\end{pmatrix},$$

$$A_0^kA_1A_0^{n-k-1})_{t=0}=\begin{pmatrix}0 & E_{n-k}\\ 0 & 0\end{pmatrix}A_1(0)\begin{pmatrix}0 & E_{k+1}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & A_k^*\\ 0 & 0\end{pmatrix},$$

where

$$A_k^*=\begin{pmatrix}a_{k+1,1}\cdots a_{k+1,k+1}\\ \cdots\cdot\\ a_{n,1}\ \cdots\ a_{n,k+1}\end{pmatrix},$$

when (j, k) element of $A_1(0)$ is denoted by a_{jk}. So, at the turning point $t=0, \psi_n(A_0)$ is a triangular matrix, i.e., all the elements below the main diagonal are zero. And the diagonal elements are accordingly eigenvalues of $\psi_n(A_0)$ at the origin, they are

$$(a_{n,1}, a_{n,1}+1, \cdots, a_{n,1}+(n-1)).$$

The coefficient matrix on the left-hand side of (4.1) is

$$H=\begin{pmatrix} -t & 0 & \cdots\cdots & n \\ 0 & (n-1)t & & 0 \\ 0 & 0 & (n-1)t & \\ & & \cdots\cdots\cdots & \\ 0 & 0 & \cdots\cdots & (n-1)t \end{pmatrix}.$$

As in the proof of Theorem 1, we substitute

$$\mu(t, \varepsilon) \sim \sum_{j=0}^{\infty} \mu_j(t)\varepsilon^j$$

into (4.1) to have an infinite sequence of systems of differential equations,

(4.2.0) $$\mu_0' H(t) = \mu_0 G(t),$$

(4.2.k) $$\mu_k' H(t) = \mu_k G(t) + \sigma_k \mu_0 + g_k(t, u),$$

where

(4.3) $$G(t) = \sigma_0 E - \psi_n(A_0) + tA_1$$

and σ_k is the constant defined in (1.9); the vector $g_k(t, \mu)$ depends linearly on $\mu_0, \cdots \mu_{k-1}$ and their derivatives. Eigenvalues of $G(0)$ are

$$\sigma_0 - a_{n,1} - (k-1) \qquad k=1, 2, \cdots, n.$$

Let us define

(4.4) $$\sigma_0 = a_{n,1}.$$

Then multiplying the matrix

$$\begin{pmatrix} -1 & 0 & \cdots & n/(n-1)t \\ 0 & 1/(n-1) & \cdots & 0 \\ & & \cdot & \\ & & \cdot & \\ 0 & 0 & \cdots & 1/(n-1) \end{pmatrix}$$

from the right to (4.2.0.), we can diagonalize this system to

(4.2.0)* $$t\mu_0' = \mu_0 G^*(t).$$

The eigenvalues of the matrix $G^*(0)$ are

$$0, -1/(n-1), \cdots, -1,$$

since the $(n, 1)$ element of $G(t)$ is identically zero.

Suppose we have obtained a sequence of holomorphic solutions $\mu_0(t), \mu_1(t), \cdots, \mu_{k-1}(t)$, then in order that the system (4.2.k) has an holomorphic solution, we must determine the constant σ_k so that the first component of

$$\sigma_k \mu_0 + g_k(t, \mu)$$

must be zero for $t=0$. This choice is possible. Indeed $G^*(0)$ has 0 as an eigenvalue, and $(1, 0, \cdots, 0)$ is the corresponding eigenvector. Consequently, there is a solution such that $\mu_0(0) = (1, 0, 0, \cdots, 0)$. This completes the proof of Theorem 2.

References

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