125. On Approximation and Uniform Approximation of Spaces

By Zdeněk FROLÍK

Mathematical Institute, Charles University, Prague, ČSSR

(Comm. by K. KUNUGI, M.J.A., Nov. 13, 1961)

All spaces under consideration are supposed to be completely regular. Concepts of approximation and uniform approximation of spaces are introduced. In Theorems 1–4 the difference between approximations and uniform approximations is shown. Finally, in Theorems 5 and 6 certain general results on approximations are stated.

Definition of approximations. Let $\boldsymbol{A}$ be a class of spaces. A space X is *approximated* by $\boldsymbol{A}$ in Y if $X\subset Y$ and for every x in $Y-X$ there exists an A in $\boldsymbol{A}$ with $X\subset A\subset Y-(x)$. The class of all spaces X which are approximated by $\boldsymbol{A}$ in $\beta(X)$ will be called the *closure* of $\boldsymbol{A}$ and will be denoted by $\mathrm{cl}(\boldsymbol{A})$.

Definition of uniform approximations. Let $\boldsymbol{A}$ be a class of spaces. A space X is *uniformly approximated* by $\boldsymbol{A}$ in Y if $X\subset Y$ and for every closed (in Y) set $F\subset Y-X$ there exists an A in $\boldsymbol{A}$ with $X\subset A\subset Y-F$. The class of all spaces X which are uniformly approximated by $\boldsymbol{A}$ in $\beta(X)$ will be called the *uniform closure* of $\boldsymbol{A}$ and denoted by unif. $\mathrm{cl}(\boldsymbol{A})$.

For convenience a class of spaces $\boldsymbol{A}$ will be called closed (uniformly closed) if $\mathrm{cl}(\boldsymbol{A})=\boldsymbol{A}$ (unif. $\mathrm{cl}(\boldsymbol{A})=\boldsymbol{A}$). From the definitions one can prove at once the following elementary formulae:

(1) $\boldsymbol{A}\subset$ unif. $\mathrm{cl}(\boldsymbol{A})\subset \mathrm{cl}(\boldsymbol{A})$

(2) $\mathrm{cl}(\mathrm{cl}(\boldsymbol{A}))=\mathrm{cl}(\boldsymbol{A})$

(3) unif. cl(unif. $\mathrm{cl}(\boldsymbol{A}))=$ unif. $\mathrm{cl}(\boldsymbol{A})$

(4) unif. $\mathrm{cl}(\boldsymbol{A}_1\smile \boldsymbol{A}_2)=$ unif. $\mathrm{cl}(\boldsymbol{A}_1)\smile$ unif. $\mathrm{cl}(\boldsymbol{A}_2)$.

Theorem 1. The uniform closure of the class $\boldsymbol{K}_\sigma$ of σ-compact spaces (countable unions of compact subspaces) is the class of all Lindelöf spaces (every open covering contains a countable subcovering).

Theorem 2. The closure of the class $\boldsymbol{K}_\sigma$ is the class of all Q-spaces (realcompact spaces in the terminology of Gillman and Jerison).

The proofs of both theorems are simple and may be left to the reader.

By a perfect mapping of X onto Y is meant a closed and continuous mapping of X onto Y such that the preimages of points are compact. One can prove the following results.

Theorems 1′ and 2′. The class $\boldsymbol{K}_\sigma$ in Theorems 1 and 2 may be replaced by each of the following classes: the class of all σ-compact locally compact spaces (=the class of all preimages under perfect mappings of open subspaces of the space of all real numbers), the class of all preimages under perfect mappings of complete metrizable spaces.

Let us recall that a space X is said to be topologically complete in the sense of Dieudonné if there exists a complete uniform structure in X generating the topology of X, or equivalently, if the finest uniformity of the space X is complete. A space X is said to be topologically complete in the sense of E. Čech if X is a G_δ in $\beta(X)$, or equivalently, if X is a G_δ is some (it follows, in any) compactification of X. (For further information see [2].)

Theorem 3. The closure of the class of all preimages under perfect mappings of complete metrizable spaces is the class of all topologically complete spaces in the sense of Dieudonné.

Theorem 4. The uniform closure of the class of all preimages under perfect mappings of complete metrizable spaces is the class of all paracompact spaces.

Let us note that a space X is the preimage under a perfect mapping of a complete metrizable space if and only if X is paracompact and topologically complete in the sense of E. Čech (for proof see [1]).

Theorem 4 is a new formulation of Theorem 4 of [1]. The proof of Theorem 3 follows from the following:

If φ is a pseudometric on a space X then there exists a metric space (X_φ, φ_A) and a continuous mapping f_φ of X onto X_φ such that

$$\varphi_A(f_\varphi(x), f_\varphi(y)) = \varphi(x, y).$$

There exists a perfect mapping g_φ of a subspace Y_φ of $\beta(X)$ onto the completion $(X_\varphi^*, \varphi_A^*)$ of (X_φ, φ_A). Of course, g_φ is the restriction of the Čech-Stone mapping h of $\beta(X)$ onto $\beta(X_\varphi^*)$ to $h^{-1}[X_\varphi^*]$. Now let $\boldsymbol{F}$ be a maximal family of subsets of X with the finite intersection property which is a Cauchy family with respect to the finest uniformity of X. In particular, $\boldsymbol{F}$ is a Cauchy family with respect to every pseudometric φ in X, and consequently, the intersection of closures in $\beta(X)$ of sets from $\boldsymbol{F}$ is a one-point set contained in the intersection of all Y_φ. If x is in the intersection of all Y_φ and if $\boldsymbol{B}$ is the family consisting of all neighborhoods of x in $\beta(X)$, then the family of all $X \cap B$, $B \in \boldsymbol{B}$, is a Cauchy family with respect to the finest uniformity of X.

Further Examples. The class of all m-Lindelöf spaces (m being an infinite cardinal number) is the uniform closure of the class of all spaces which are unions of m their compact subspaces. The class of all normal spaces is uniformly closed but not closed.

Now we shall state two general theorems.

Theorem 5. Let $\boldsymbol{A}$ be a class of spaces such that the preimages under perfect mappings of spaces from $\boldsymbol{A}$ belong to $\boldsymbol{A}$. Then the closure of $\boldsymbol{A}$ is closed under operation of topological product. If, moreover, closed subspaces of spaces from $\boldsymbol{A}$ belong to $\boldsymbol{A}$, then both closure of $\boldsymbol{A}$ and uniform closure of $\boldsymbol{A}$ possess the same property.

The first assertion of Theorem 5 is not true for the uniform closure (consider the class of all paracompact spaces).

Theorem 6. Let $\boldsymbol{A}$ be a class of spaces such that both images and preimages under perfect mappings of spaces from $\boldsymbol{A}$ belong to $\boldsymbol{A}$. The following conditions on a space X are equivalent:

(a) X belongs to the uniform closure of $\boldsymbol{A}$, i.e., X is uniformly approximated by $\boldsymbol{A}$ in $\beta(X)$.

(b) X is approximated by $\boldsymbol{A}$ in every compactification of X.

(c) X is uniformly approximated by $\boldsymbol{A}$ in some compactification of X.

It should be noticed that (4) is not true for closures. If in the definition of approximation one-point sets are replaced by finite sets then (4) will remain true for closures.

References

[1] Z. Frolík: On the topological product of paracompact spaces, Bull. Acad. Pol., VIII, 747-750 (1960).

[2] ——: Topologically complete spaces, Comm. Math. Univ. Carolinae 1, 3, 3-15 (1960).