142. *Evolutional Equations of Parabolic Type*

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1. **Introduction.** The object of this note is to state some theorems concerning the existence and the uniqueness of the solution of the initial value problem for the evolutional equation

$$dx(t)/dt=A(t)x(t)+f(t),\ a\leqq t\leqq b. \tag{1.1}$$

Here the unknown $x(t)$ as well as the inhomogeneous term $f(t)$ is a function on the closed interval $[a, b]$ to a Banach space X, whereas $A(t)$ is a function on $[a, b]$ to the set of (in general unbounded) linear operators acting in X.

For each $t, A(t)$ is assumed to be the infinitesimal generator of an analytic semi-group of bounded operators. We make some additional assumptions on the resolvents of $A(t)$. However, we do not assume that the domain of some fractional power of $A(t)$ is independent of t.

Under these assumptions, we will construct the *evolution operator* (or *fundamental solution*) $U(t, s)$, defined for $a\leqq s\leqq t\leqq b$, such that the solution of (1.1) can be expressed in the form

$$x(t)=U(t, s)x(s)+\int_s^t U(t, \sigma)f(\sigma)d\sigma. \tag{1.2}$$

2. **Notations and assumptions.** We denote by $\sum$ the closed angular domain consisting of all the complex numbers λ satisfying $|\arg\lambda|\leqq\pi/2+\theta$, where θ is a fixed angle with $0<\theta<\pi/2$. We make the following assumptions.

(A.1) For each $t\in[a, b], A(t)$ is a densely defined, closed linear operator whose resolvent set $\rho(A(t))$ contains $\sum$.

(A.2) There exists a positive constant M such that the resolvent of $A(t)$ satisfies

$$||(\lambda I-A(t))^{-1}||\leqq M/|\lambda|, \tag{2.1}$$

for each $t\in[a, b]$ and $\lambda\in\sum$.

(A.3) $A(t)^{-1}$, which is a bounded operator valued function of t, is once Hölder continuously differentiable in $a\leqq t\leqq b$:

$$||dA(t)^{-1}/dt-dA(s)^{-1}/ds||\leqq K|t-s|^{\alpha},\ K, \alpha>0. \tag{2.2}$$

(A.4) There exist positive constants N and ρ with $0\leqq\rho<1$, such that

$$\left\|\frac{\partial}{\partial t}(\lambda I-A(t))^{-1}\right\|\leqq\frac{N}{|\lambda|^{1-\rho}}, \tag{2.3}$$

for each $t\in[a, b]$ and $\lambda\in\sum$.

In what follows, we denote by C constants which depend only on the constants appearing in the above assumptions.

As a sufficient condition for (A.4), we have

THEOREM 1. *If there exist positive numbers ρ, ρ_1 and a natural number l satisfying $1=l\rho+\rho_1, 0\leqq\rho_1<\rho<1$, such that both $A(t)^{-\rho}$ and $A(t)^{-\rho_1}$ are once continuously differentiable in t, then* (A.4) *is satisfied.*

PROOF. This follows from

$$(\partial/\partial t)(\lambda I-A(t))^{-1}=-A(t)(\lambda I-A(t))^{-1}dA(t)^{-1}/dtA(t)(\lambda I-A(t))^{-1}, \quad (2.4)$$

$$A(t)^{-1}=(A(t)^{-\rho})^{l}A(t)^{-\rho_1}, \quad (2.5)$$

$$\|A(t)^{\alpha}(\lambda I-A(t))^{-1}\|\leqq C_{\alpha}/|\lambda|^{1-\alpha}, 0\leqq\alpha\leqq1. \quad (2.6)$$

Let Γ be any smooth contour running from $\infty e^{-i(\theta+\frac{\pi}{2})}$ to $\infty e^{i(\theta+\frac{\pi}{2})}$ in $\sum$. By the assumptions made above, each $A(s)$ generates a semi-group $\exp(tA(s))$ by means of the formula

$$\exp(tA(s))=\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda t}(\lambda I-A(s))^{-1}\,d\lambda, \quad (2.7)$$

which is analytic in the sector $|\arg t|<\theta$.

3. Construction of the evolution operator. In this section, we construct the evolution operator $U(t,s)$. Setting

$$U(t,s)=\exp((t-s)A(t))+\int_s^t\exp((t-\tau)A(t))R(\tau,s)d\tau, \quad (3.1)$$

and then calculating formally, we are led to the integral equation for $R(t,s)$:

$$R(t,s)-\int_s^t R_1(t,\tau)R(\tau,s)d\tau=R_1(t,s), \quad (3.2)$$

where $R_1(t,s)=-(\partial/\partial t+\partial/\partial s)\exp((t-s)A(t))$. The integral representation for $R_1(t,s)$ is

$$R_1(t,s)=-\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda(t-s)}\frac{\partial}{\partial t}(\lambda I-A(t))^{-1}d\lambda, \quad (3.3)$$

hence by (2.3),

$$\|R_1(t,s)\|\leqq C(t-s)^{-\rho}. \quad (3.4)$$

Using (3.4), the integral equation (3.2) can be solved by successive approximation, and $R(t,s)$ satisfies

$$\|R(t,s)\|\leqq C(t-s)^{-\rho}. \quad (3.5)$$

LEMMA 1. *For $s<\tau<t$, we have*

$$\|R(t,s)-R(\tau,s)\|$$
$$\leqq C\left\{\frac{t-\tau}{(t-s)(\tau-s)^{\rho}}+\frac{(t-\tau)^{\alpha}}{t-s}+\frac{(t-\tau)^{1-\rho}}{(t-s)^{\rho}}+\frac{(t-\tau)^{\alpha}}{(t-s)^{\rho}}\log\frac{t-s}{t-\tau}\right\}. \quad (3.6)$$

PROOF. First, we write as

$$R_1(t,s)-R_1(\tau,s)=-\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda(t-s)}\Big(\frac{\partial}{\partial t}(\lambda I-A(t))^{-1}-\frac{\partial}{\partial\tau}(\lambda I-A(\tau))^{-1}\Big)d\lambda$$
$$-\frac{1}{2\pi i}\int_{\Gamma}(e^{\lambda(t-s)}-e^{\lambda(\tau-s)})\frac{\partial}{\partial\tau}(\lambda I-A(\tau))^{-1}d\lambda. \quad (3.7)$$

By (2.4), (2.2) and (2.3), we get

$$\left\|\frac{\partial}{\partial t}(\lambda I-A(t))^{-1}-\frac{\partial}{\partial \tau}(\lambda I-A(\tau))^{-1}\right\| \leqq C\{(t-\tau)|\lambda|^{\rho}+(t-\tau)^{\alpha}\}. \tag{3.8}$$

Using this, we obtain

$$\|R_1(t, s)-R_1(\tau, s)\| \leqq C\left(\frac{t-\tau}{(t-s)(\tau-s)^{\rho}}+\frac{(t-\tau)^{\alpha}}{t-s}\right). \tag{3.9}$$

The inequality (3.6) follows from (3.2), (3.5) and (3.9) (cf. Lemma 1.2 in [4]). If we write

$$W(t, s)=\int_s^t \exp((t-\tau)A(t))R(\tau, s)d\tau, \tag{3.10}$$

we have

$$\begin{aligned}\frac{\partial}{\partial t}W(t, s)=&\int_s^t \frac{\partial}{\partial t}\exp((t-\tau)A(t))(R(\tau, s)-R(t, s))\,d\tau\\ &-\int_s^t R_1(t, \tau)d\tau\cdot R(t, s)+\exp((t-s)A(t))R(t, s).\end{aligned} \tag{3.11}$$

Using (3.6), we easily obtain

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\| \leqq \frac{C}{t-s}, \quad \left\|\frac{\partial}{\partial t}W(t, s)\right\| \leqq C\left(\frac{1}{(t-s)^{\rho}}+\frac{1}{(t-s)^{1-\alpha}}\right). \tag{3.12}$$

Similarly, $A(t)U(t, s)$ and $A(t)W(t, s)$ are also bounded operators if $s<t$, and satisfy similar inequalities to (3.12).

Next, we must prove that the solution of (1.1) is *uniquely* determined by the initial data and the inhomogeneous term. For this purpose, it is sufficient to show that there exists a bounded operator valued function $V(t, s)$ with the property:

i) $V(t, s)$ is defined and strongly continuous in $a \leqq s \leqq t \leqq b$,

ii) $V(s, s)=I$ for each $s \in [a, b]$,

iii) for any $x \in D(A(s))$, $\lim_{h \to 0} h^{-1}\{V(t, s+h)-V(t, s)\}x$ exists and equals to $-V(t, s)A(s)x$.

Such $V(t, s)$ is constructed by putting

$$V(t, s)=\exp((t-s)A(s))+\int_s^t Q(t, \tau)\exp((\tau-s)A(s))ds, \tag{3.13}$$

and it turns out that $V(t, s)=U(t, s)$ as a corollary (cf. [3], p. 146). Summing up, we have

THEOREM 2. *Under* (A.1)~(A.4), *there exists an evolution operator* $U(t, s)$ *for* (1.1) *which satisfies*

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\|=\|A(t)U(t, s)\| \leqq \frac{C}{t-s}. \tag{3.14}$$

For any $x \in X$ *and any Hölder continuous* $f(t)$,

$$x(t)=U(t, s)x+\int_s^t U(t, \sigma)f(\sigma)d\sigma$$

represents the unique solution of (1.1) *in* $s<t\leqq b$ *satisfying the initial condition* $x(s)=x$.

4. Analyticity of the solution.

THEOREM 3. *If* $A(t)^{-1}$ *is holomorphic in* t *in a complex neigh-*

bourhood $\triangle$ *of* $[a, b]$ *and* (A.1)$\sim$(A.4) *are satisfied in* $\triangle$ (*not only in* $[a, b]$), *the solution* $x(t)$ *of* (1.1) *is holomorphic in* t *in any subdomain of* $\triangle$ *where* $f(t)$ *is holomorphic.*

PROOF. It is sufficient to prove that $U(t, s)$ is holomorphic in $s, t \in \triangle, |\arg(t-s)| < \theta_1$ for some positive θ_1. This can be achieved quite similarly as in Komatsu [3].

If $A(t)^{-h}$ is holomorphic and h^{-1} is some natural number, then $A(t)^{-1}$ is also holomorphic. Hence, Theorem 3 is a generalization of Theorem 2 of Kato [1].

5. Perturbation theory. In this section, we consider a perturbed equation

$$dx(t)/dt = A(t)x(t) + B(t)x(t) + f(t). \tag{5.1}$$

(A.5) For each $t \in [a, b]$, $B(t)$ is a closed operator such that $D(B(t)) \supset D(A(t))$. Moreover, for any $t \in [a, b]$ and $\lambda \in \sum$,

$$\| B(t)(\lambda I - A(t))^{-1} \| \leqq H|\lambda|^{\gamma-1},$$

where H and γ are positive constants such that $0 \leqq \gamma < 1$.

(A.6) $B(t)A(t)^{-1}$ is Hölder continuous:

$$\| B(t)A(t)^{-1} - B(s)A(s)^{-1} \| \leqq L|t-s|^{\beta},$$

where L and β are some positive constants.

THEOREM 4. *Under* (A.1)$\sim$(A.6), *there exists an evolution operator* $U(t, s)$ *for* (5.1). $U(t, s)$ *satisfies the similar inequalities to* (3.14). $U(t, s)$ *also satisfies* i)$\sim$iii) *in* 2, *hence the solution of* (5.1) *is uniquely determined by its initial data and the inhomogeneous term.*

6. Remark. Professor T. Kato indicated the following remark:

Let $X = L^2(0, 1) = \Big\{$the set of all square integrable functions $x(\xi)$ with the norm $\|x\| = \left(\int_0^1 |x(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}\Big\}$, and $A(t) = -|\xi - t|^{-k}$, $\xi > 1$.

Then (A.1)$\sim$(A.4) are all satisfied. Especially, (2.3) holds good with $\rho = k^{-1}$. But, $A(t)^{-\rho}$ is differentiable only if $\rho k \geqq 1$. Hence, if k is sufficiently near 1, the assumption of Theorem 1 is not satisfied. Thus, Theorem 1 does not give a very satisfactory sufficient condition for the validity of (2.3). Notice that the domain of $A(t)^h$ does change with t for any $h > 0$ in this example.

References

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