22. On a Fourier Invariant Distribution Space

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§1. According to the theory of distributions by L. Schwartz, the Fourier transform of any tempered distribution (element of the space (S')) is defined as another tempered distribution. [1] Now a question arises from this fact. Let $(\boldsymbol{D}')$ denote the space of distribution defined by L. Schwartz. Let $(\boldsymbol{Z}')$ denote the space which is obtained from $(\boldsymbol{D}')$ by Fourier transform. [4] The question is the following:

"Is the space (S') the furthest Fourier invariant distribution space in the space $(\boldsymbol{D}')$, or not?" In another word, "Is there any distribution space which is invariant with respect to Fourier transform, is contained in the space $(\boldsymbol{D}')$ and contains the space (S'), or not?" [2]

In this paper we show an affirmative answer to this question in the following manner:

In §2 we define the space $(\boldsymbol{D}')\cap(\boldsymbol{Z}')$ which includes the space (S') and does not equal to (S'). This space is invariant for Fourier transform and is contained in the space $(\boldsymbol{D}')$. In §3 we construct an element of the space $(\boldsymbol{D}')\cap(\boldsymbol{Z}')\cap(S')^c$. In §4 we construct an element of the space $(\boldsymbol{D}')\cap(\boldsymbol{Z}')\cap(S')^c$ which is an entire function.

§2. The notations and the definitions.

Let $\mathbf{L}_\alpha$ denote a linear complete topological space, and let $\tau(\mathbf{L}_\alpha)$ denote its topology.

Let $(\boldsymbol{D})$ and $(\boldsymbol{S})$ denote the function spaces defined by L. Schwartz. [1]

Let $(\boldsymbol{Z})$, $\mathbf{S}_\alpha$, $\mathbf{S}^\beta$ and $\mathbf{S}_\alpha^\beta$ denote the function spaces defined by Gelfand and Silov. [4] Namely the space $(\boldsymbol{Z})$ is the Fourier transform of the space $(\boldsymbol{D})$, and the other spaces are defined as follows:

$$\mathbf{S}_\alpha=\{\varphi;\ |x^k\varphi^{(q)}(x)|\leq C_q A^k k^{k\alpha},\ \varphi\in(\boldsymbol{S})\},$$

$$\mathbf{S}^\beta=\{\varphi;\ |x^k\varphi^{(q)}(x)|\leq C_k B^q q^{q\beta},\ \varphi\in(\boldsymbol{S})\},$$

and

$$\mathbf{S}_\alpha^\beta=\{\varphi;\ |x^k\varphi^{(q)}(x)|\leq CA^k B^q k^{k\alpha} q^{q\beta},\ \varphi\in(\boldsymbol{S})\}$$

where the constants A, B, C, C_q and C_k depend on φ.

Let $\mathbf{S}_0^\infty$ denote the function space $\mathbf{S}_0^\infty=\bigcup_\beta \mathbf{S}_0^\beta$ and let $\mathbf{S}_\infty^0$ denote the function space $\mathbf{S}_\infty^0=\bigcup_\alpha \mathbf{S}_\alpha^0$. About the exact definitions of $\mathbf{S}_\alpha$, $\mathbf{S}^\beta$, $\mathbf{S}_\alpha^\beta$, $\mathbf{S}_0^\infty$, and $\mathbf{S}_\infty^0$, see [4] and [6].

Let $\tau(A)<\tau(B)$ denote that the topology of the space B is stronger than the topology of the space A.

Let $\tau(A)=\tau(B)$ denote that the topology of B is equivalent to the topology of A.

We can see easily that the following Lemma holds.

Lemma 1. *If* $\mathbf{L}_\alpha$ *satisfies the following relations;*

(1) $(\boldsymbol{D})\subseteq\mathbf{L}_\alpha\subseteq(\boldsymbol{S})$, (1′) $\tau((\boldsymbol{D}))\geqq\tau(\mathbf{L}_\alpha)\geqq\tau((\boldsymbol{S}))$,

(2) $(\boldsymbol{Z})\subseteq\mathbf{L}_\alpha\subseteq(\boldsymbol{S})$, (2′) $\tau((\boldsymbol{Z}))\geqq\tau(\mathbf{L}_\alpha)\geqq\tau((\boldsymbol{S}))$,

then $\mathbf{L}_\alpha$ *has the following properties;*

(a) $(\boldsymbol{D}')\supseteq\mathbf{L}'_\alpha\supseteq(\boldsymbol{S}')$, (b) $(\boldsymbol{Z}')\supseteq\mathbf{L}'_\alpha\supseteq(\boldsymbol{S}')$,

(c) *The Fourier transform* $\mathfrak{F}(\mathbf{L}_\alpha)$ *of the space* $\mathbf{L}_\alpha$ *satisfies the condition* (1), (1′), (2), (2′), *where the topology* $\tau(\mathfrak{F}(\mathbf{L}_\alpha))$ *is given similar as in* [1], [4].

Moreover if $\mathbf{L}_\alpha$ *also satisfies the following conditions;*

(3) $(\boldsymbol{D})$ *is dense in the space* $\mathbf{L}_\alpha$,

(4) $(\boldsymbol{Z})$ *is dense in the space* $\mathbf{L}_\alpha$,

then $\mathfrak{F}(\mathbf{L}_\alpha)$ *also satisfy the conditions* (3), (4).

Definition 1. $\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_B=\bigcup_\alpha\mathbf{L}'_{\alpha}$, *where* $\bigcup_\alpha\mathbf{L}'_\alpha$ *is the join of all spaces which satisfy the conditions* (1), (1′), (2), (2′), (3) *and* (4) *in* **Lemma 1.**

Definition 2. *Let* $\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_F$ *denote all the sequences of distributions* ($\in(\boldsymbol{S}')$) *which are convergent in the topology* $(\boldsymbol{D}')$ *and in the topology* $(\boldsymbol{Z}')$, *too.*

We understand **Definition 1** *or* **Definition 2** *as the definition of* $(\boldsymbol{D}')\cap(\boldsymbol{Z}')$.

The precise meaning of the space $(\boldsymbol{D}')\cap(\boldsymbol{Z}')\cap(\boldsymbol{S}')^c$ in §1 is $\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_B\cap(\boldsymbol{S}')^c\cap\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_F$.

§3. According to the **Theorem 2** in [6], there exists a function φ_0 such that belongs to the space $\mathbf{S}_0$ and does not belong to $\mathbf{S}_0^\infty$. We can assume this function φ_0 has carrier in $[0,1]$ without loss of generality. Using this function φ_0 we construct the following function ψ_0 and distribution T_0.

(1) The construction of the sequence.

Since φ_0 does not belong to $\mathbf{S}_0^\infty$, the inequality $|x^k\varphi_0^{(q)}(x)|\leq CA^kB^qq^{\beta q}$, $(k,q=0,1,2,\cdots)$ is not satisfied. Namely for all fixed A, B, C, β, there exist integers k, q such that $\mathrm{Max}_x|x^k\varphi^{(q)}(x)|\geq CA^kB^qq^{\beta q}$. Taking $A>1$, it follows that $\mathrm{Max}_x|\varphi_0^{(q)}(x)|\geq CB^qq^{\beta q}$.

Now we select two sequences $\{B_i\}$ and $\{\beta_i\}$ which satisfy the following relations:

$$\beta_1<\beta_2<\beta_3<\cdots,\quad \lim_{i\to\infty}\beta_i=\infty$$
$$B_1<B_2<B_3<\cdots,\quad \lim_{i\to\infty}B_i=\infty.$$

Corresponding these sequences we select sequence $\{q_i\}$ defined by the equality $q_i=q_i(\beta_i,B_i)=\mathrm{Min}_q\{q;\mathrm{Max}_x|\varphi^{(q)}(x)|>CB_i^qq^{q\beta_i}\}$.

We construct the sequence $\{a_q\}$ $(q=1,2,\cdots)$ by the following way: $a_q=\mathrm{Min}\{1/\mathrm{Max}|\varphi^{(q)}(x)|,\ 1/CB_{i+1}^qq^{q\beta_{i+1}}\}$ for $q_i<q\leq q_{i+1}$.

We denote $\sqrt{a_q}$ by α_q, $(q=0,1,2,\cdots)$.

(2) Let T_0 be the distribution $\sum_{q=0}^{\infty}\alpha_q\tau_{2q}\delta^q$, where τ_{2q} is the translation of the length $2q$ to the positive sense.

(3) Let $\psi_0=\sum_{q=0}^{\infty}\alpha_q\tau_{s(q)}\varphi_0$ where the length $s(q)$ of the translation is decided by the condition such that $|\langle(\tau_{2q}\delta^{(q)}),(\tau_{s(q)}\varphi_0)\rangle|$ takes the maximum value.

(4) Let $\mathbf{L}_{T_0}$ denote the space of the functions $\varphi\in(S)$ which satisfy the condition $|\lim_{n\to\infty}T_0^n\varphi|<+\infty$, where $T_0^n=\sum_{q=0}^{n}\alpha_q\tau_{2q}\delta^{(q)}$.

(5) Let $V(k, \mathrm{m}, \varepsilon_1, \varepsilon_2, \{T_0^n\})$ denote the neighbourhood in $\mathbf{L}_{T_0}$ which satisfy the conditions; $V\in\varphi$ means $|\lim_{n\to\infty}T_0^n\varphi|<\varepsilon_1$ and $|(1+r^2)^kD^p\varphi(x)|<\varepsilon_2$ for any order of derivations $|p|<m$.

We easily see that $\mathbf{L}_{T_0}$ is a complete linear topological space, $(S)\supseteq\mathbf{L}_{T_0}\supseteq(\boldsymbol{D})$ and $(S)\supseteq\mathbf{L}_{T_0}\supseteq(\boldsymbol{Z})$.

We are now ready to some Lemmas.

Lemma 2. *$(\boldsymbol{D})$ is dense in the space $\boldsymbol{L}_{T_0}$.*

Proof. Let $\beta_n(x)$ denote the following functions; $0\leqq\beta_n(x)\leqq1$, $\beta_n(x)\in C^\infty$ and $\beta_n(x)=\begin{cases}0 & \text{for } |x|\geq n+1\\ 1 & \text{for } |x|\leq n\end{cases}$.

Let f be a function in the space $\mathbf{L}_{T_0}$. Then $\beta_n(x)f\in(\boldsymbol{D})$ and $\lim_{n\to\infty}\beta_n(x)f=f$ in $\mathbf{L}_{T_0}$. (Q.E.D.)

Lemma 3. *$(\boldsymbol{Z})$ is dense in the space $\mathbf{L}_{T_0}$.*

Proof. Take a function $\varphi\in\mathbf{S}_\alpha^0\subseteq(\boldsymbol{Z})$ $(\alpha>1)$ which satisfy the condition $\int\varphi(x)\,dx=\Phi\neq0$, then ψ satisfies the following inequalities;

$$|x^k\varphi^{(q)}(x)|\leqq CB^qA^kk^{k\alpha}\ \text{for}\ q=0,1,2,\cdots,k=0,1,2,\cdots.$$

For any function $f\in(\boldsymbol{D})$, we construct the sequence $\{f_m(x)\}$ by the following form; $f_m=f(x)*\varphi(mx)/m\Phi$ $(m=1,2,3,\cdots)$. Then there exist a positive constant M, a positive integer K which has the property $f(x)=0$ for $|x|\geq K$, and a sequence of positive number $\{\varepsilon_m\downarrow0\}$ such that the following inequality is satisfied,

$$|\lim_{n\to\infty}\langle T_0^n,f_m-f\rangle|=|\lim_{n\to\infty}\langle\textstyle\sum_{q=0}^{n}\tau_{2q}\alpha_q\delta^{(q)},\,f(x)*\varphi(mx)/m\Phi-f\rangle|$$
$$\leqq\textstyle\sum_{n=k}^{\infty}\mathrm{Min}_k\,MCB^nA^kk^{k\alpha}m^{n-1}\alpha_n/\Phi\{m(n-K+1)\}^k+\varepsilon_mK.$$

The foregoing inequality takes the following form for $k=2n$; $|\lim_{n\to\infty}\langle T_0^n,f_m-f\rangle|\leqq MCB^nA^{2n}(2n)^{2n\alpha}m^{n-1}\alpha_n/\Phi\{m(n-K+1)\}^{2n}+\varepsilon_mK$. So we can see that the following equality $\lim_{m\to\infty}|\lim_{n\to\infty}\langle T_0^n,f_m-f\rangle|=0$. Using this result and **Lemma 2,** we see easily that **Lemma** 3 holds.

Remark: *In §4 we construct an example T_0 which is an entire function. We obtain* **Lemma 2, 3** *also in such a case.*

Lemma 4. *T_0 belongs to the space $(\boldsymbol{D}')$ and $(\boldsymbol{Z}')$.*

Proof. Since T_0 is a locally finite sum of $\delta^{(q)}$ $(q=0,1,2,\cdots)$, T_0 belongs to $(\boldsymbol{D}')$.

Let $\mathfrak{B}$ be a bounded set of $(\boldsymbol{Z})$, then we can express $\mathfrak{B}$ by the following form; $\mathfrak{B}=\bigcap_{k,q}\{\psi; |x^k\psi^{(q)}(x)|\leqq C_k B^q\}$, where B is a constant. So, $|\psi^{(q)}(x)|\leqq C_0 B^q$ for $q=0,1,2,\cdots$. For constant B, there exists number B_{i+1} such that the inequality $2B<\sqrt{B_{i+1}}$ holds.

For any q which is larger than q_i,

the inequality $\alpha_q C_0 B^q\leqq\sqrt{C_0}/2^q\sqrt{q^{q\beta_{i+1}}}$ holds.

So $\alpha_q C_0 B^q<1/2^q$ for sufficiently large q.

Hence $|\langle T_0,\psi\rangle|=|\langle\sum_{q=0}^{\infty}\alpha_q\tau_{2q}\delta^{(q)},\psi\rangle|\leqq\sum_{q=0}^{\infty}\alpha_q C_0 B^q<K<\infty$.

Since in the space $(\boldsymbol{Z}')$, a linear bounded functional is a linear continuous functional similarly in the space $(\boldsymbol{D}')$, T_0 belongs to the space $(\boldsymbol{Z}')$. [1]

From **Lemma 4**, we see that $\mathbf{L}_{T_0}$ satisfies the condition (1′) (2′) in **Lemma 1**.

Lemma 5. *The function* ψ_0 *belongs to the space* $(\boldsymbol{S})$.

Proof. For any integer k, there exist B_{i+1} and β_{i+1} such that both the inequalities $B_{i+1}>2^{4k}$ and $\beta_{i+1}>2k$ are satisfied.

For any fixed integer k and any q such that satisfy $q>\text{Max}\,(2,q_i)$, we can see the following inequality holds;

$$|\alpha_k(2q+2)^k|\leqq(2q+2)^k/\sqrt{CB_{i+1}^q q^{q\beta_{i+1}}}\,(\underset{q\to\infty}{\sim}1/\sqrt{C}).$$

On the other hand we see

$$\lim_{x\to\infty}|x^k\psi_0^{(p)}(x)|=\lim_{x\to\infty}|x^k\alpha_q(x)\varphi_0^{(p)}(x)|\leqq\lim_{x\to\infty}\left|x^k\alpha_q(x)\,\text{Max}_x|\varphi_0^{(p)}(x)|\right|,$$

where $\alpha_q(x)\begin{cases}=\alpha_q & \text{for } 0\leq x-2q<2 \ (q \text{ non negative integer})\\ =0 & \text{for } x<0.\end{cases}$

We can also see $\left|x^k\alpha_q(x)\,\text{Max}_x|\varphi_0^p(x)|\right|\leq|(2q+2)^k\alpha_q|M(P)$

for $2q\leq x<2q+2$, where $M(P)=\text{Max}_x|\varphi_0^{(p)}(x)|$.

Hence

$$\lim_{x\to\infty}|x^k\psi_0^{(p)}(x)|\leq\lim_{q\to\infty}|(2q+2)^{k+1}\alpha_q|M(P)/(2q+2)\leq\lim_{q\to\infty}1/\sqrt{C}(2q+2)=0.$$

So $\psi_0(x)$ belongs to the space $(\boldsymbol{S})$.

Lemma 6. $\langle T_0,\psi_0\rangle$ *does not converge.*

Proof. $\langle T_0,\psi_0\rangle=\langle\{\sum_{q=0}^{\infty}\alpha_q\tau_{2q}\delta^{(q)}\},\{\sum_{q=0}^{\infty}\alpha_q\tau_{s(q)}\varphi_0\}\rangle=\sum_{q=0}^{\infty}\langle\alpha_q\tau_{2q}\delta^{(q)},\tau_{s(q)}\varphi_0\rangle$. Since $\langle\alpha_{q_i}\tau_{2q_i}\delta^{(q_i)},\tau_{s(q_i)}\varphi_0\rangle=1$, $\langle T_0,\psi_0\rangle$ does not converge.

Theorem 1. $(\boldsymbol{S}')\subsetneqq(\boldsymbol{D}')\cap(\boldsymbol{Z}')$.

Proof. From **Lemmas 4, 5, 6,** $T_0\in\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_F\cap(\boldsymbol{S}')^c$. From **Lemmas 1, 2, 3, 4, 5, 6,** $T_0\in\{(\boldsymbol{D}')\cap(\boldsymbol{Z}')\}_B\cap(\boldsymbol{S}')^c$. Hence $T_0\in(\boldsymbol{D}')\cap(\boldsymbol{Z}')\cap(\boldsymbol{S}')^c$.

§4. Hereafter we use the following entire function;

$$\rho_k(x)=\exp\{-(kx)^2\}/K, \text{ where } K=\int_{-\infty}^{\infty}\exp\{-(kx)^2\}\,dx.$$

Let $\{k_i; i=0,1,2,\cdots\}$ denote the integer sequence which has the following properties;

(1) $\operatorname{Max}|\partial_x^{(i)}\{\varphi_0(x)*\rho_{k_i}(x)\}|\geq(1-(1/(i+2)^2))\operatorname{Max}|\partial_x^{(i)}\varphi(x)|$,

(2) $k_i>\{(i+3)!\}^2 i^i$,

(3) $|\partial_x^{(i)}\{\varphi_0(x)*\rho_{k_i}(x)\}|\leq 1/n^2(i+2)^2$ for $|x|>n$.

Let's construct the series $E(x)=\sum_{q=0}^{\infty}\alpha_q\tau_{2q}(\delta^{(q)}*\rho_{k_q}(x))=\sum_{q=0}^{\infty}\alpha_q\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x))$.

Lemma 7. *$E(x)$ is an entire function.*

Proof. $\partial_x^{(q)}\rho_{k_q}(x)=\partial_x^{(q)}\exp\{-(k_q x)^2\}/K=\partial_x\{\partial_x^{(q-1)}\exp\{-(k_q x)^2\}/K\}$

$=\partial_x\{P_{q-1}(x)\exp\{-(k_q x)^2\}\}=\{\partial_x P_{q-1}(x)-2k_q^2 xP_{q-1}(x)\}\exp\{-(k_q x)^2\}$,

where $P_{q-1}(x)$ is a polynomial whose order is $q-1$.

So, from the property (2) of k_i we see

$$|\partial_x^{(q)}\rho_{k_q}(x)|\leq 2^q(k_q)^{2q}q!\,|x^q|\exp\{-(k_q x)^2\}\text{ for }|x|\geq 2.$$

If $(x+iy)$ is contained in an arbitrary compact set A, then

$$\sum_{q=0}^{\infty}|\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x+iy))|\leq\sum_{q=0}^{K}|\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x+iy))|$$
$$+\sum_{q=K+1}^{\infty}2^q(k_q)^{2q}q!\,(2q)^q|1-((x+iy)/2q)|^q\exp\{-k_q^2[(x-2q)+iy]^2\}$$
$$\leq\sum_{q=0}^{K}|\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x+iy))|+\sum_{q=K+1}^{\infty}(k_q)^{2q}q!\,q^q4^q\exp\{-k_q^2(2q)^2/2\}$$

for sufficiently large K.

Using the Stirling's formula, we see

$$\sum_{q=K+1}^{\infty}(k_q)^{2q}q!\,q^q4^q\exp\{-k_q^2(2q)^2/2\}<\sum_{q=K+1}^{\infty}(2k_q q)^{2q}\exp\{-k_q^2(2q)^2/2\}$$

for sufficiently large K, q.

Since $\log[(2k_q q)^{2q}\exp\{-2k_q q)^2/2\}]=2q\log(2k_q q)-((2k_q q)^2/2)$

$=2q[\log(2k_q q)-(k_q(2k_q q)/2)]<-q^2$ for sufficiently large q.

$$\sum_{q=K+1}^{\infty}(2k_q q)^{2q}\exp\{-(2k_q q)^2/2\}<\sum_{q=K+1}^{\infty}\exp(-q^2)$$

for sufficiently large K.

Hence we can see that

$$\sum_{q=0}^{\infty}|\alpha_q\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x+iy))|\leq\sum_{q=0}^{k}|\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x+iy))|$$
$$+\sum_{q=K+1}^{\infty}\exp(-q^2)<+\infty.$$

Since a series of entire functions $\sum_{q=0}^{\infty}\alpha_q\tau_{2q}(\delta^{(q)}*\rho_{k_q}(x))$ converges uniformly in an arbitrary compact set, $E(x)$ is an entire function.

Lemma 8. *$E(x)$ belongs to the space $(\boldsymbol{D}')\cap(\boldsymbol{Z}')$.*

Proof. We easily see that $E(x)$ belongs to $(\boldsymbol{D}')$, since $E(x)$ is an entire function.

We prove next that $E(x)$ belongs to $(\boldsymbol{Z}')$.

$$\langle E(x),\psi(x)\rangle=\langle\sum_{q=0}^{\infty}\alpha_q\tau_{2q}(\partial_x^{(q)}\rho_{k_q}(x)),\psi(x)\rangle$$
$$=\langle\sum_{q=0}^{\infty}\alpha_q\delta^{(q)}*\rho_{k_q}(x),\tau_{-2q}\psi(x)\rangle=\langle\sum_{q=0}^{\infty}\alpha_q\delta^{(q)}(\rho_{k_q}(x)*\tau_{-2q}\psi(x))\rangle,$$

for $\psi(x)\in(\boldsymbol{Z})$.

Using the property (1) of k_i for $\psi\in B$ in **Lemma 4,** we see that the following inequality $|\partial_x^{(q)}\rho_{k_q}(x)*\tau_{-2q}\psi(x)|=|\rho_{k_q}(x)*\tau_{-2q}\psi^{(q)}(x)|\leq C_0B^q$ holds.

Hence by the same way as **Lemma 4** in §3, we see that $E(x)\in(\boldsymbol{Z}')$.

Lemma 9. *$E(x)$ does not belong to $(\boldsymbol{S}')$.*

Proof. Consider the inner product $\langle E(x),\psi_0\rangle$ for the function ψ_0 in §3, then we see that the following equality

$$\langle E(x),\psi_0\rangle=\langle\sum_{q=0}^{\infty}\alpha_q\tau_{2q}\delta^{(q)}*\rho_{k_q},\sum_{q=0}^{\infty}\alpha_q\tau_{s(q)}\varphi_0\rangle$$
$$=\sum_{q=0}^{\infty}\alpha_q^2<\tau_{2q}\delta^{(q)}*\rho_{k_q},\tau_{s(q)}\varphi_0\rangle$$
$$+\sum_{q}\sum_{q\neq q'}\alpha_q\alpha_{q'}\langle\tau_{2q}\delta^{(q)}*\rho_{k_q},\tau_{s(q')}\psi_0\rangle\text{ holds.}$$

From the property (3) of k_i, we see that $|\alpha_q \alpha_{q'}|$ are uniformly bounded for any q q', and $|\sum_{q'=q} \langle \tau_{2q}\delta^{(q)} * \rho_{k_q}, \tau_{s(q')}\varphi_0 \rangle| \leq 1/(q+2)^2 \sum_{i=1}^{\infty} 1/i^2$.

Hence $|\sum_q \sum_{q' \neq q} \alpha_q \alpha_{q'} \langle \tau_{2q}\delta^{(q)} * \rho_{k_q}, \tau_{s(q)}\varphi_0 \rangle| < C < +\infty$.

For $q = q_i$, $|\alpha_q^2 \langle \tau_{2q}\delta^{(q)} * \rho_{k_q}, \tau_{s(q)}\varphi_0 \rangle|$

$$\geqq \left| \mathrm{Max}\, |\varphi_0^{(q)}(x)| (1 - (1/(q+2)^2)) / \mathrm{Max}\, |\varphi_0^{(q)}(x)| \right| = (1 - (1/(q+2)^2)).$$

So the series $\sum_{q=0}^{\infty} \alpha_q^2 \langle \tau_{2q}\delta^{(q)} * \rho_{k_q}, \tau_{s(q)}\varphi_0 \rangle$ does not converge.

Hence $\langle E(x), \psi_0 \rangle$ does not converge i.e. $E(x)$ does not belong to (S').

Theorem 2. *The space $(D') \cap (Z') \cap (S')^c$ contains an entire function.*

Proof. From **Lemmas 7, 8, 9,** $E(x) \in \{(D') \cap (Z')\}_F \cap (S')^c$. By the same method as in **Lemmas 2, 3,** and by the result of **Lemmas 7, 8, 9,** $E(x) \in \{(D') \cap (Z')\}_B \cap (S')^c$.

Hence $E(x) \in (D') \cap (Z') \cap (S')^c$.

Lemma 10. $|\mathfrak{F}\rho_k(x)| \leq 1$.

Proof. $|\mathfrak{F}\rho_k(x)| = \left| \int \exp(ixs)\rho_k(x)\, dx \right| = \left| \int \exp(ixs - k^2x^2)/K\, dx \right|$

$$\leqq \int |\exp(ixs - k^2x^2)/K|\, dx = \int \exp(-k^2x^2)/K\, dx = 1.$$

Lemma 11. $|\mathfrak{F} \cdot \tau_{2q}(\delta^{(q)} * \rho_{k_q}(x))| \leq |s|^q$.

Proof. $|\mathfrak{F} \cdot \tau_{2q}(\delta^{(q)} * \rho_{k_q}(x))| = |\mathfrak{F} \cdot (\tau_{2q}\delta^{(q)} * \rho_{k_q}(x))| \leq |\mathfrak{F} \cdot \tau_{2q}\delta^{(q)}| \cdot |\mathfrak{F}\rho_{k_q}(x)|$
$\leq |(is)^q \cdot e^{i \cdot 2qs}| = |s|^q$.

Theorem 3. *The space $(D') \cap (Z') \cap (S')^c$ contains an entire function $E(x)$ whose Fourier transform $\mathfrak{F}E(x)$ is also an entire function.*

Proof. Let's consider the function $E(x) = \sum_{q=0}^{\infty} \alpha_q \tau_{2q}(\partial_x^{(q)} \rho_{k_q}(x))$. From **Theorem 2,** $E(x)$ is an entire function. Since $1/\alpha_q \geq q!$ for sufficiently large q, it follows from **Lemmas 10, 11** that $\mathfrak{F}E(x)$ is also an entire function.

$E(x)$ and $\mathfrak{F}E(x)$ belong to $(D') \cap (Z') \cap (S')^c$.

Hence the consequence of **Theorem 3** is obtained.

References

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