20. A Remark on Mapping Spaces

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1. Let X and Y be topological spaces. We shall consider the set of all continuous mappings of X into Y. This set is turned into a topological space by the compact-open topology; this topology is defined by selecting as a sub-basis for the open sets the family of sets $T(K, G)$, where K ranges over all the compact subsets of X and G ranges over all the open subsets of Y and $T(K, G)$ denotes the set of all continuous mappings f of X into Y such that $f(K)\subset G$. As usual, we write Y^X for the *mapping space* thus obtained.

Now let X and Y be Hausdorff spaces, and let Z be a topological space. Then, with any continuous mapping f of $X\times Y$ into Z, there is associated a mapping f^* from Y to the mapping space Z^X by the formula

$$[f^*(y)](x)=f(x, y).$$

The correspondence $f\to f^*$ defines a one-to-one mapping

$$\theta\colon Z^{X\times Y}\to(Z^X)^Y.$$

K. Morita has shown in [2] that this mapping θ is always a homeomorphism into. But it is not necessarily a homeomorphism onto. We shall consider Hausdorff spaces, for which the mapping θ is a homeomorphism onto for any locally compact Hausdorff space Y and any topological space Z, and denote the class of such spaces by Θ.

In [2], the following notion has been introduced. A Hausdorff space X will be said to *have the weak topology with respect to compact sets in the wider sense* if a subset A of X such that $A\frown K$ is closed for every compact subset K of X is necessarily closed. We shall denote by $\mathfrak{B}$ the class of Hausdorff spaces having the weak topology with respect to compact sets in the wider sense. It is known that locally compact spaces and CW-complexes in the sense of J.H.C. Whitehead belong to the class $\mathfrak{B}$. Moreover, it has been shown in [2, Theorem 4] that $\mathfrak{B}\subset\Theta$, but it is unknown to the author whether $\mathfrak{B}=\Theta$ or not.

Now, let X be a topological space and $\{A_\alpha\}$ a closed covering of X. The space X is said to *have the weak topology with respect to* $\{A_\alpha\}$ *in the wider sense* if any subset of X whose intersection with each A_α is closed is necessarily closed. Then, the main result of this note is stated as follows.

Theorem 1. *Let X be a Hausdorff space having the weak* *topology with respect to a closed covering* $\{A_\alpha\}$ *in the wider sense. If each* A_α *belongs to the class* Θ, *then* X *also belongs to the class* Θ.

As a corollary, we have the following.

Corollary 1. *Let* X *be a Hausdorff space which is the inductive limit space* ([3]) *of the sequence of spaces* $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots$. *If each* X^n *belongs to the class* Θ, *then* X *belongs to the class* Θ.

As an application, we have the following theorem.

Theorem 2. *Let* (X, C) *be a relative CW-complex in the sense of G. W. Whitehead* ([4]). *If the subspace* C *belongs to the class* Θ, *then the space* X *belongs to the class* Θ.

This result is useful in the obstruction theory of relative CW-complexes.

2. Proof of Theorems. Lemma 1. *Let* X *be a topological space having the weak topology with respect to a closed covering* $\{A_\alpha\}$ *in the wider sense. Then a mapping* f *of* X *into another topological space* Y *is continuous if and only if* $f | A_\alpha$ *is continuous for each* A_α.

Lemma 2. *Let* X *be a topological space having the weak topology with respect to a closed covering* $\{A_\alpha\}$ *in the wider sense. If* Y *is a locally compact Hausdorff space, the product space* $X \times Y$ *has the weak topology with respect to the closed covering* $\{A_\alpha \times Y\}$ *in the wider sense.*

These lemmas have been proved in [1].

Now let g be any element of $(Z^X)^Y$, and A a closed subset of X. We define a mapping g_A from Y to the mapping space Z^A by the formula $g_A(y) = g(y) | A$.

Lemma 3. $g_A \in (Z^A)^Y$.

Proof. Let y be any point of Y. Let $T'(K, G) = \{f \mid f \in Z^A, f(K) \subset G\}$ be any open subset such that $g_A(y) \in T'(K, G)$, where K is a compact subset of A and G an open subset of Z. Since K is also a compact subset of the space X, the open subset $T(K, G)$ of Z^X is defined. Moreover $[g(y)](x) = [g_A(y)](x) \in G$ for each $x \in K$ since $K \subset A$. Thus $T(K, G)$ is a neighborhood of $g(y)$, and since $g \in (Z^X)^Y$, there exists a neighborhood U of y such that $g(U) \subset T(K, G)$. Then, for each $z \in U$, we have $[g_A(z)](x) = [g(z)](x) \in G$ for every $x \in K$. Therefore, $g_A(U) \subset T'(K, G)$, and lemma is established.

Proof of Theorem 1. According to [2, Theorem 1], the mapping θ is a homeomorphism into for the space X.

We shall show that θ is onto. Let g be any continuous mapping of Y into Z^X. We define a mapping $\tilde{g}$ from $X \times Y$ to Z by $\tilde{g}(x, y) = [g(y)](x)$, where $x \in X, y \in Y$. On the other hand, by Lemma 3, $g_{A_\alpha} \in (Z^{A_\alpha})^Y$, and since $A_\alpha \in \Theta$ by assumption, there is an element $\tilde{g}_\alpha \in Z^{A_\alpha \times Y}$ such that $\theta(\tilde{g}_\alpha) = g_{A_\alpha}$. But $\tilde{g}_\alpha(x, y) = [g_{A_\alpha}(y)](x) = [g(y)](x)$ for $(x, y) \in A_\alpha \times Y$. Hence $\tilde{g}_\alpha = \tilde{g} | A_\alpha \times Y$ and thus $\tilde{g} | A_\alpha \times Y$ is continuous for each α. Therefore $\tilde{g}$ is continuous by Lemmas 1 and 2. Clearly $\theta(\tilde{g})=g$, and this proves that θ is onto.

Our theorem is thus established.

Proof of Theorem 2. Let (X, C) be a relative CW-complex, and $\{E_\alpha^n \mid \alpha \in J_n, n=0,1,2,\cdots\}$ its relative CW-decomposition. We put $X^n = C \smile (\smile_{m \leqq n} \smile_{\alpha \in J_m} E_\alpha^m)$. At first, we show that X^0 belongs to the class θ.

For any element f of $(Z^{X^0})^Y$, we define a mapping $\tilde{f}$ from $X^0 \times Y$ to Z by $\tilde{f}(x, y) = [f(x)](y)$, where $x \in X^0$, $y \in Y$. Then, as in the proof of Theorm 1, $f \mid C \times Y$ and $f \mid E_\alpha^0 \times Y$ are continuous for each α. Since X^0 has the weak topology with respect to the closed covering $\{C\} \smile \{E_\alpha^0 \mid \alpha \in J_0\}$, $\tilde{f}$ is a continuous mapping by Lemmas 1 and 2. Clearly $\theta(\tilde{f})=f$. Hence θ is onto, and since θ is a homeomorphism into by [2, Theorem 1], we have $X^0 \in \Theta$.

We proceed by induction. Assume that X^{n-1} belongs to the class Θ. Since X^n is the adjunction space obtained by adjoining $\{I_\alpha^n \mid \alpha \in J_n\}$ to the space X^{n-1}, X^n has the weak topology with respect to the closed covering $\{X^{n-1} \smile E_\alpha^n \mid \alpha \in J_n\}$ according to [3, Lemma 2]. Since each E_α^n is a compact subspace and hence belongs to the class Θ, each subspace $X^{n-1} \smile E_\alpha^n$ belongs to the class Θ. Therefore X^n belongs to the class Θ by Theorem 1.

Finally, since X is, as has been shown in [3, Lemma 3], the inductive limit space of the sequence $X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots$, and each X^n belongs to the class Θ, X belongs to the class Θ by Corollary 1. Thus, our theorem is established.

3. Notes on the class $\mathfrak{B}$. Theorem 3. *Let X be a topological space and A its closed subspace. If X belongs to the class $\mathfrak{B}$, then A also belongs to the class $\mathfrak{B}$.*

Proof. Let B be any subset of A such that $B \frown K$ is closed for every compact subset K of A. Let L be any compact subset of X. Then $A \frown L$ is a compact subset of A and hence $B \frown L = (B \frown A) \frown L = B \frown (A \frown L)$ is closed by assumption. Since X belongs to the class $\mathfrak{B}$, B is closed, which proves the Theorem 3.

Remark 1. When X belongs to the class Θ, it remains unknown whether its any closed subspace A belongs to the class Θ or not. But, if any continuous mapping from A to Z has an extension over X, A also belongs to the class Θ. For instance, if X is a normal space, or A is a retract of X, then A belongs to the class Θ.

Theorem 4. *Let X be a Hausdorff space having the weak topology with respect to a closed covering $\{A_\alpha\}$ in the wider sense. Then X belongs to the class $\mathfrak{B}$ if and only if each subspace A_α belongs to the class $\mathfrak{B}$.*

Proof. Suppose that each A_α belongs to the class $\mathfrak{B}$. Let A be any subset of X such that $A\frown K$ is closed for every compact subset K of X. Then, for any compact subset L of A_α, we have $(A\frown A_\alpha)\frown L =(A\frown L)\frown A_\alpha$, which is closed since L is a compact subset of X. As A_α belongs to the class $\mathfrak{B}$, $(A\frown A_\alpha)$ is closed. Hence A is a closed subset of X by assumption, and thus X belongs to the class $\mathfrak{B}$.

The "only if" part is obtained from Theorem 3.

Corollary 2. *Let X be a Hausdorff space which is the inductive limit space of a sequence of spaces $X^1\subset X^2\subset X^3\subset\cdots\subset X^n\subset\cdots$. Then X belongs to the class $\mathfrak{B}$ if and only if each space X^n belongs to the class $\mathfrak{B}$.*

Theorem 5. *Let X be a Hausdorff space which is the adjunction space obtained by adjoining $\{C_\alpha\}$ to C by means of the continuous mappings $\{g_\alpha: A_\alpha\to C\}$ ([3]). If C and each C_α belong to the class $\mathfrak{B}$, then X also belongs to the class $\mathfrak{B}$.*

Proof. Let $W=C\smile(\underset{\alpha}{\smile}C_\alpha)$ be the disjoint union, and $p: W\to X$ the natural projection. Let A be any subset of X such that $A\frown K$ is closed for every compact subset K of X. Then, it is sufficient to show that $p^{-1}(A)$ is a closed subset of W.

Let L be any compact subset of C_α. Then

$$(p^{-1}(A)\frown C_\alpha)\frown L=p^{-1}(A)\frown L=p^{-1}(A\frown p(L))\frown L,$$

and since $p(L)$ is a compact subset of Z, $A\frown p(L)$ is closed by assumption. Hence $(p^{-1}(A)\frown C_\alpha)\frown L$ is closed. But C_α belongs to the class $\mathfrak{B}$, and so $p^{-1}(A)\frown C_\alpha$ is a closed subset of C_α. Similarly $p^{-1}(A)\frown C$ is also a closed subset of C. Therefore $p^{-1}(A)$ is a closed subset of W by definition, and Theorem 5 is established.

Remark 2. Let X be a space belonging to the class $\mathfrak{B}$, and f a continuous mapping from X onto a space Y. If we can prove that Y also belongs to the class $\mathfrak{B}$, then Theorem 5 is obtained as a consequence of Theorem 4 according to [3, Lemma 3]. But it seems to be false in general that Y belongs to the class $\mathfrak{B}$, and so we proved Theorem 5 independently.

Theorem 6. *Let (X, C) be a relative CW-complex. Then X belongs to the class $\mathfrak{B}$ if and only if the subspace C belongs to the class $\mathfrak{B}$.*

The "if" part follows from Theorem 5 and Corollary 2 according to [3, Lemma 3]. The "only if" part follows from Theorem 3.

Remark 3. In his recent paper "Homotopy Theory in General Categories" (Math. Annalen, 144 (1961)), P. J. Huber has stated that the following result is proved in the Mimeographed Note in 1960 by Eckmann and Huber:

If X is a Hausdorff k-space (J. L. Kelley, General Topology, p. 230) and Δ_n the Euclidean n-simplex, then two spaces $Map'(X, Map(\Delta_n, Y))$ and $Map(\Delta_n, Map'(X, Y))$ are identified, where $Map(X, Y)$

means the mapping space Y^X and $Map'(X, Y)$ denotes the subspace of $Map(X, Y)$, consisting of the basepoint preserving mappings.

But K. Morita had already proved in 1956 a more general theorem ([2, Theorem 6]):

If the product space $X\times Y$ *belongs to the class* $\mathfrak{B}$, *then* θ *induces a homeomorphism onto*:

$$\theta_0\colon (Z, Z_0)^{(X\times Y,\, X_0\times Y\cup X\times Y_0)}\to((Z, Z_0)^{(X, X_0)}, Z_0^X)^{(Y, Y_0)},$$

where $X_0\subset X$, $Y_0\subset Y$ *and* $Z_0\subset Z$.

References

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