19. On Infinitesimal Operators of Irreducible Representations of the Lorentz Group of n-th Order

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§1. **Introduction.** The Lorentz group of n-th order is the connected component of the identity element of the group of such homogeneous linear transformations in the real n-dimensional vector space that leave the quadratic form $x_1^2+x_2^2+\cdots+x_{n-1}^2-x_n^2$ invariant. We shall denote it by L_n.

In the present paper, we investigate differentiable irreducible representations by bounded (not necessarily unitary) operators in a Hilbert space. We shall make use of what is called the infinitesimal method.

First we establish the system of commutation relations which must be fulfilled by the corresponding infinitesimal operators. Next we give a class of the solutions of these operator equations. It is believed that there exist no other solutions, but the proof of this fact is not completed. In another paper [6] we shall classify irreducible representations and distinguish unitary ones.

The same problem has been discussed for the case $n=5$ by L. H. Thomas [1] and T. D. Newton [2] and for the case $n=4$ by M. A. Naimark [3]. The results in the present paper and in [6] may throw light on the problem of explicite construction of irreducible unitary representations of these groups and suggest the existence of integrable irreducible unitary representations when n is odd.

§2. **Lie algebra of L_n.** Consider in L_n $n(n-1)/2$ one-parameter subgroups of the following types:

$$g_{ij}(t)=\begin{array}{c} \qquad (i) \qquad\quad (j) \\ \left\|\begin{matrix} 1 & & & & \\ & \ddots & & & \\ & & \cos t\cdots\sin t & & \\ & & \cdots\cdots\cdots & & \\ & & -\sin t\cdots\cos t & & \\ & & & \ddots & \\ & & & & 1\end{matrix}\right\|\end{array}, \quad g_k(t)=\begin{array}{c} \qquad\qquad (k) \\ \left\|\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & \cosh t\cdots\sinh t \\ & & 0\ \cdots\ 0 \\ & & \cdots\cdots\cdots \\ & & \sinh t\cdots\cosh t\end{matrix}\right\|\end{array}, \tag{1}$$

where $1\leq i, j\leq n-1, 1\leq k\leq n-1$.

The matrix $g_{ij}(t)$ corresponds to a rotation in the plane (x_ix_j) and the matrix $g_k(t)$ corresponds to a hyperbolic rotation in the plane (x_kx_n).

A maximal compact subgroup U_n of L_n is generated by all $g_{ij}(t)$ and it is isomorphic with the special orthogonal group of $(n-1)$-th order $SO(n-1)$.

Put $a_{ij}=\frac{d}{dt}g_{ij}(t)|_{t=0}$ and $b_k=\frac{d}{dt}g_k(t)|_{t=0}$, then they form a basis of the Lie algebra of L_n, if we take only a_{ij} for which $i>j$. The commutation relations are as follows:

$$[a_{i_1j_1}, a_{i_2j_2}]=\delta_{i_1j_2}a_{j_1i_2}+\delta_{j_1i_2}a_{i_1j_2}-\delta_{i_1i_2}a_{j_1j_2}-\delta_{j_1j_2}a_{i_1i_2},$$
$$[b_{k_1}, b_{k_2}]=a_{k_1k_2}$$

and

$$[a_{ij}, b_k]=\delta_{jk}b_i-\delta_{ik}b_j. \tag{2}$$

§3. Commutation relations of infinitesimal operators. A weakly continuous representation $\{T_g, \mathfrak{H}\}$ of L_n is called *differentiable* if there exists in $\mathfrak{H}$ a dense invariant subspace $\mathfrak{D}$ which has the following properties. (a) All infinitesimal operators corresponding to one-parameter subgroups and their products are defined in $\mathfrak{D}$. (b) Let $a(t)$ be any one-parameter subgroup and A be its infinitesimal operator. $T_{a(t)}$ are expressed in $\mathfrak{D}$ in the form:

$$T_{a(t)}f=\sum_{p=0}^{\infty}\frac{1}{p!}A^pf,\ f\in\mathfrak{D}. \tag{3}$$

For differentiable representations, the infinitesimal operators A_{ij}, B_k corresponding respectively to a_{ij} and b_k satisfy in $\mathfrak{D}$ the similar commutation relations to (2). (4)

From the relations (4), it is easy to see that a representation is completely determined when we know the operators $A_{21}, A_{32},\cdots, A_{n-1,n-2}$ and B_{n-1}, because other operators can be expressed by them by means of the relations (4). Moreover we can prove that it is necessary and sufficient to pick up the following $n(n+1)/2-4$ relations:

$$\begin{aligned}&[A_{i,i-1}, A_{j,j-1}]=0, \qquad i-1>j,\\ &[[A_{i,i-1}, A_{i-1,i-2}], A_{i,i-1}]=A_{i-1,i-2},\\ &[[A_{i,i-1}, A_{i-1,i-2}], A_{i-1,i-2}]=-A_{i,i-1};\end{aligned} \tag{5}$$

$$\begin{aligned}&[B_{n-1}, A_{i,i-1}]=0, \qquad i\leq n-2,\\ &[A_{n-1,n-2}, [B_{n-1}, A_{n-1,n-2}]]=B_{n-1},\\ &[B_{n-1}, [B_{n-1}, A_{n-1,n-2}]]=A_{n-1,n-2}.\end{aligned} \tag{6}$$

Then the remaining relations are automatically satisfied.

Similar facts can be proved more easily for $SO(n)$. The close relation between Lie algebras of these two groups plays an important role in solving the above operator equations (5) and (6).

If we know the operators $A_{21}, A_{32},\cdots, A_{n-1,n-2}$, then it is sufficient to investigate the only one operator B_{n-1} and $(n-1)$ relations (6).

§4. **Determination of infinitesimal operators.** In the following we consider only differentiable representations. Generally we denote an equivalent class of irreducible representation of U_n by β.

For a representation $\{T_g, \mathfrak{H}\}$ of L_n, the Hilbert space $\mathfrak{H}$ is a direct sum of subspaces $\mathfrak{H}^\beta$, each of which provides an irreducible representation of U_n. In each $\mathfrak{H}^\beta$ the operators $A_{21}, A_{32}, \cdots, A_{n-1,n-2}$ can be explicitely written down by the result of I. M. Gelfand and M. L. Cejtlin for $SO(n-1)$ (see [4] or [5]).

A representation $\{T_g, \mathfrak{H}\}$ of L_n is called *irreducible* if the following two conditions are satisfied: (1) There exists no closed subspace which is invariant with respect to all operators T_g. (2) The bounded operators that commute with all T_g are only constant multiples of the identity operator.

Now we study possible irreducible representations $\{T_g, \mathfrak{H}\}$ of L_n which satisfy the next assumption (U) (which may probably be proved for any irreducible representations):

(U) $\{T_g, \mathfrak{H}\}$ *contains each irreducible representation of* U_n *at most once.*

In $\mathfrak{H}^\beta$ a basis may be so chosen, uniquely up to a constant factor, that the induced representation of U_n is in a given canonical form. The basis of $\mathfrak{H}$ consisted of these bases is determined uniquely up to constant factors which depend only on β, because $\mathfrak{H}$ contains a given $\mathfrak{H}^\beta$ at most once under the assumption (U).

Therefore the operators $A_{21}, \cdots, A_{n-1,n-2}$ are completely determined and the remaining one B_{n-1} is to be determined from the relations (6). We shall indicate one type of B_{n-1}, without detailed computation. This is the only type of B_{n-1} for the cases $n=3, 4$ and 5 and it is believed that this fact holds also for the cases $n\geq 6$.

There are remarkable differences according the parity of n.

I. The case of odd $n: n=2k+1\,(k=1, 2, \cdots)$. Consider the diagrams

$$\lambda=\begin{bmatrix} m_{2k-1,1} \; m_{2k-1,2} \cdots\cdots\cdots\cdots m_{2k-1,k} \\ m_{2k-2,1} \cdots\cdots\cdots m_{2k-2,k-1} \\ m_{2k-3,1} \cdots\cdots\cdots m_{2k-3,k-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_{41} \qquad\quad m_{42} \\ m_{31} \qquad\quad m_{32} \\ m_{21} \\ m_{11} \end{bmatrix}$$

where the numbers m_{ij} are integers which obey the conditions

$$|m_{2k-1,1}|\leq m_{2k-1,2}\leq\cdots\leq m_{2k-1,k} \tag{7}$$

and for $p=1, 2, \cdots, k-1$,

$$\begin{aligned} &|m_{2p+1,1}|\leq m_{2p,1}\leq m_{2p+1,2}, \\ &m_{2p+1,i} \leq m_{2p,i}\leq m_{2p+1,i+1}, \qquad i=2,\cdots,p, \\ &-m_{2p,1} \leq m_{2p-1,1}\leq m_{2p,1}, \\ &m_{2p,i-1} \leq m_{2p-1,i}\leq m_{2p,i}, \qquad i=2,\cdots,p. \end{aligned} \tag{8}$$

The first row of λ determines an irreducible representation of U_n, therefore it may be identified with the notation β. For a fixed β the remaining m_{ij} run through the numbers which satisfy the condition (8) and to each λ corresponds a vector $\xi(\lambda)$ of $\mathfrak{H}^\beta$. These vectors $\xi(\lambda)$ form a basis of $\mathfrak{H}^\beta$ and letting β run through the appropriate values they form a basis of $\mathfrak{H}$. We write the formulas for operators $A_{2p+1,2p}, A_{2p+2,2p+1}$ and B_{n-1} in the basis $\xi(\lambda)$. Denote by λ_r^{j+} (λ_r^{j-}) the diagram obtained from λ in replacing m_{rj} by $m_{rj}+1$ ($m_{rj}-1$ respectively).

The formulas for the operators $A_{2p+1,2p}, A_{2p+2,2p+1}$ have the form

$$A_{2p+1,2p}\xi(\lambda)=i\sum_{j=1}^{p}A_{2p-1}^{j}(\lambda)\xi(\lambda_{2p-1}^{j+})+i\sum_{j=1}^{p}A_{2p-1}^{j}(\lambda_{2p-1}^{j-})\xi(\lambda_{2p-1}^{j-}),$$

$$A_{2p+2,2p+1}\xi(\lambda)=i\sum_{j=1}^{p}B_{2p}^{j}(\lambda)\xi(\lambda_{2p}^{j+})+i\, C_{2p}(\lambda)\xi(\lambda)+i\sum_{j=1}^{p}B_{2p}^{j}(\lambda_{2p}^{j-})\xi(\lambda_{2p}^{j-}). \tag{9}$$

The coefficients A_{2p-1}^{j}, B_{2p}^{j} and C_{2p} are experessed as follows: Putting

$$l_{2p,j}=m_{2p,j}+j,\ l_{2p-1,j}=m_{2p-1,j}+(j-1),\quad (j=1,2,\cdots,p), \tag{10}$$

$$A_{2p-1}^{j}(\lambda)= \tag{11}$$

$$=\frac{1}{2}\left[\frac{\prod_{r=1}^{p-1}[(l_{2p-2,r}-1/2)^2-(l_{2p-1,j}+1/2)^2]\prod_{r=1}^{p}[(l_{2p,r}-1/2)^2-(l_{2p-1,j}+1/2)^2]}{\prod_{r\neq j}(l_{2p-1,r}^2-l_{2p-1,j}^2)[l_{2p-1,r}^2-(l_{2p-1,j}+1)^2]}\right]^{\frac{1}{2}};$$

$$B_{2p}^{j}(\lambda)=\left[\frac{\prod_{r=1}^{p}(l_{2p-1,r}^2-l_{2p,j}^2)\prod_{r=1}^{p+1}(l_{2p+1,r}^2-l_{2p,j}^2)}{l_{2p,j}^2(4l_{2p,j}^2-1)\prod_{r\neq j}(l_{2p,r}^2-l_{2p,j}^2)[(l_{2p,r}-1)^2-l_{2p,j}^2]}\right]^{\frac{1}{2}},$$

$$C_{2p}(\lambda)=\frac{\prod_{r=1}^{p}l_{2p-1,r}\prod_{r=1}^{p+1}l_{2p+1,r}}{\prod_{r=1}^{p}l_{2p,r}(l_{2p,r}-1)}. \tag{12}$$

The operator B_{n-1} is given in the form

$$B_{n-1}\xi(\lambda)=\sum_{j=1}^{k}A^{j}(\lambda)\xi(\lambda_{2k-1}^{j+})+\sum_{j=1}^{k}A^{j}(\lambda_{2k-1}^{j-})\xi(\lambda_{2k-1}^{j-}). \tag{13}$$

Here the coefficients A^j are given as follows: Introduce a row of integers $\alpha=(n_1, n_2,\cdots, n_{k-1})$ satisfying the condition

$$0\leq n_1\leq n_2\leq\cdots\leq n_{k-1} \tag{14}$$

and a complex number c, then putting $l_r=n_r+r-1/2, (1\leq r\leq k-1)$,

$$A^{j}(\lambda)=\frac{1}{2}\times \tag{15}$$

$$\times\left[\frac{\prod_{r=1}^{k-1}[(l_{2k-2,r}-1/2)^2-(l_{2k-1,j}+1/2)^2][l_r^2-(l_{2k-1,j}+1/2)^2]\cdot[c^2-(l_{2k-1,j}+1/2)^2]}{\prod_{r\neq j}(l_{2k-1,r}^2-l_{2k-1,j}^2)[l_{2k-1,r}^2-(l_{2k-1,j}+1)^2]}\right]^{\frac{1}{2}}.$$

II. The case of even n: $n=2k+2\,(k=1,2,\cdots)$. The diagram has the form

$$\lambda=\begin{bmatrix} m_{2k,1} & m_{2k,2} & \cdots\cdots\cdots\cdots & m_{2k,k} \\ m_{2k-1,1} & m_{2k-1,2} & \cdots\cdots\cdots\cdots & m_{2k-1,k} \\ & m_{2k-1,1} \quad m_{2k-2,2} & \cdots & m_{2k-2,k-1} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & & \\ & m_{41} \qquad m_{42} & & \\ & m_{31} \qquad m_{32} & & \\ & m_{21} & & \\ & m_{11} & & \end{bmatrix}$$

where the numbers m_{ij} are integers and satisfy the conditions

$$0\leq m_{2k,1}\leq m_{2k,2}\leq\cdots\leq m_{2k,k} \tag{16}$$

and similar inequalities to (8).

The situation is quite similar with the case I.

Only the operator B_{n-1} is different. It is given in the form

$$B_{n-1}\xi(\lambda)=\sum_{j=1}^{k}B^{j}(\lambda)\xi(\lambda_{2k}^{j+})+C(\lambda)\xi(\lambda)+\sum_{j=1}^{k}B^{j}(\lambda_{2k}^{j-})\xi(\lambda_{2k}^{j-}). \tag{17}$$

The coefficients B^{j} and C are given as follows. Introduce as in the case I $\alpha=(n_1, n_2,\cdots, n_k)$ and c. Here the inequality corresponding to (14) is

$$|n_1|\leq n_2\leq\cdots\leq n_k, \tag{18}$$

and

$$B_{2k}^{j}(\lambda)=\left[\frac{\prod_{r=1}^{k}(l_{2k-1,r}^2-l_{2k,j}^2)\prod_{r=1}^{k}(l_r^2-l_{2k,j}^2)\cdot(c^2-l_{2k,j}^2)}{l_{2k,j}^2(4l_{2k,j}^2-1)\prod_{r\neq j}(l_{2k,r}^2-l_{2k,r}^2)[(l_{2k,r}-1)^2-l_{2k,j}^2]}\right]^{\frac{1}{2}},$$

$$C_{2k}(\lambda)=\frac{\prod_{r=1}^{k}l_{2k-1,r}\prod_{r=1}^{k}l_r\cdot c}{\prod_{r=1}^{k}l_{2k,r}(l_{2k,r}-1)}, \tag{19}$$

where $l_r=n_r+(r-1),\,(1\leq r\leq k)$.

We shall discuss in [6] the classification of irreducible representations which have the infinitesimal operators of the above type.

References

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