17. *Singularities of the Solution of a Non-linear Wave Equation*

By Sadakazu AIZAWA
Department of Mathematics, Kōbe University
(Comm. by K. KUNUGI, M.J.A., March 12, 1962)

1. Concerning the characteristic initial value problem for the non-linear wave equation

$$(1)\qquad \square u \equiv \frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 u}{\partial x_3^2}=f(x_1, x_2, x_3, u)$$

in two space variables, the author [1] proved the existence of a generalized solution satisfying vanishing initial condition under the assumption that i) $f(x,u)$ is continuous in $\mathfrak{D}$, and ii) the inequality $\square\underline{\omega}(x)\leq f(x,u)\leq\square\overline{\omega}(x)$ holds in $\mathfrak{D}$, where

$$\mathfrak{D}=\{(x,u);\, x\in\overline{D},\, \underline{\omega}(x)\leq u\leq\overline{\omega}(x)\}.^{1)}$$

In this note we are concerned with singularities of the solution of the characteristic initial value problem for (1), that is, we shall show in §3 that for a certain class of functions $f(x,u)$, the solution of (1) with vanishing initial condition becomes infinite at a (finite) point in D. The proof is similar to that given by J. B. Keller [2] for the solution of the Dirichlet problem concerning the non-linear elliptic equation $\Delta u=f(u)$.

2. To give an explicit bound on the solution of (1), we shall make use of the following comparison theorem.

THEOREM 1. *Let $f(x,u)$ and $\underline{f}(x,u)$ be continuous functions defined for $x\in\overline{D}$ and $-\infty<u<+\infty$, and let the inequality $f(x,u)\geq\underline{f}(x,\underline{u})$ hold for $x\in\overline{D}$ and $u\geq\underline{u}$. Further let $\underline{f}(x,u)$ be Lipschitz continuous*[2] *with respect to u.*

Assume that $u(x)$ and $\underline{u}(x)$ are generalized solutions in D of the equations $\square u=f(x,u)$, $\square\underline{u}=\underline{f}(x,\underline{u})$ with initial conditions $u(x)=\varphi(x)$, $\underline{u}(x)=\underline{\varphi}(x)$ on S_Σ respectively, where $\varphi(0)\geq\underline{\varphi}(0)$ and $\partial\varphi/\partial\lambda_x\geq\partial\underline{\varphi}/\partial\lambda_x$ on S_Σ. Then the inequality $u(x)\geq\underline{u}(x)$ holds in $\overline{D}$.

For the proof, see Theorem 2.3 in [1].

3. We begin by considering the ordinary non-linear differential equation of the second order

$$(2)\qquad \frac{d^2v}{dr^2}+\frac{\lambda}{r}\frac{dv}{dr}=h(r,v)\quad(\lambda>0)$$

1) For the notation refer to S. Aizawa: Differentiability of the generalized solution of a non-linear wave equation, Proc. Japan Acad., **38**, 69–74 (1962).

2) $|\underline{f}(x,u_1)-\underline{f}(x,u_2)|\leq L(M)\,|u_1-u_2|$ provided $|u_1|,\ |u_2|\leq M$.

with initial conditions

$$v(0)=v_r(0)=0. \tag{3}$$

LEMMA 1. *Let $h(r,v)$ be a non-negative continuous function defined for $0\leq r<+\infty$ and $-\infty<v<+\infty$. Further let $h(r,v)$ be Lipschitz continuous with respect to u. Then a unique solution of* (2) *and* (3) *exists in the interval $(0,\alpha)$. If α is finite, then the solution $v(r)$ tends to infinity as $r\to\alpha$.*

Proof. Rewriting (2) in the form

$$(r^\lambda v_r)_r=r^\lambda h(r,v) \tag{4}$$

and integrating (4) from 0 to r, we have

$$v_r(r)=r^{-\lambda}\int_0^r x^\lambda h(x,v(x))\,dx. \tag{5}$$

From (5) it follows that $v_r(r)\geq 0$. Hence $v(r)$ is a non-decreasing function of r, which proves our assertion. The uniqueness is obvious.

We say that a continuous function $f(x,u)$ defined for $x\in\bar{D}$ and $-\infty<u<+\infty$ satisfies *Condition* (*G*) in D if there exists a continuous function $h(r,v)$ defined for $0\leq r<+\infty$ and $-\infty<v<+\infty$ and satisfying the following conditions:

i) $h(r,v)$ is non-decreasing, that is, $h(r_1,v_1)\geq h(r_2,v_2)$ provided $r_1\geq r_2$ and $v_1\geq v_2$,

ii) $h(r,v)$ is Lipschitz continuous with respect to v,

iii) $h(r,v)\geq 0$ for $v\geq 0$,
$h(r,v)>0$ for $r>0$ and $v\geq 0$,

iv) $f(x,u)\geq h(r,u)$, where we set $r=(x_1^2-x_2^2-x_3^2)^{\frac{1}{2}}$,

v) for any fixed $r>0$,

$$\int_0^\infty\Big[\int_0^x h(r,z)\,dz\Big]^{-\frac{1}{2}}dx<+\infty,$$

vi) D contains a point x whose (Lorentzian) distance r_x from the origin is greater than

$$\sqrt{\frac{3}{2}}\int_{v_0}^\infty\Big[\int_{v_0}^x h(r_0,z)\,dz\Big]^{-\frac{1}{2}}dx+r_0$$

for some $r_0>0$, where $v_0=v(r_0)$ and $v(r)$ is the solution of (2) and (3).

Now we can prove the following theorem.

THEOREM 2. *If $f(x,u)$ satisfies Condition (G) in D, then the solution of* (1) *with vanishing initial condition becomes infinite at a (finite) point in D.*

Proof. It is obvious that if we set $r=(x_1^2-x_2^2-x_3^2)^{\frac{1}{2}}$ and $\lambda=2$ in (2), the solution $v(r)$[3] of (2) and (3) becomes a unique solution[4] of the equation $\Box v=h(r,v)$ with vanishing condition. Moreover, it

3) From iii) it follows that $v(r)$ is not an identically zero function.

4) The uniqueness follows from Theorem 1.

is clear from ii), iii) and Theorem 1 that $v(r)\geq 0$. Hence it follows from (5) (with $\lambda=2$) that $v_r(r)\geq 0$. Accordingly, $v(r)$ is a monotone increasing function of r. From (5), we have in virtue of i)

$$v_r(r)\leq (r/3)h(r, v(r)).$$

Now, from (2)

$$v_{rr}\geq h(r, v(r))/3. \tag{6}$$

Multiplying (6) by v_r and integrating from r_0 to r, we have in virtue of i)

$$v_r(r)^2-v_r(r_0)^2\geq \frac{2}{3}\int_{r_0}^{r} h(x, v(x))v_r(x)\, dx$$

$$\geq \frac{2}{3}\int_{r_0}^{r} h(r_0, v(x))v_r(x)\, dx=\frac{2}{3}\int_{v_0}^{v} h(r_0, z)\, dz,$$

where $v_0=v(r_0)$. Hence

$$v_r(r)^{-1}\leq\sqrt{\frac{3}{2}}\Big[\int_{v_0}^{v} h(r_0, z)\, dz\Big]^{-\frac{1}{2}}.$$

Integrating again from r_0 to r, we get

$$r-r_0\leq\sqrt{\frac{3}{2}}\int_{v_0}^{v}\Big[\int_{v_0}^{x} h(r_0, z)\, dz\Big]^{-\frac{1}{2}} dx. \tag{7}$$

By v) the integral in (7) converges as $v\to+\infty$. Hence it follows from (7) that $v(r)$ becomes infinite at a finite value of r. Moreover, by vi), there is a point in D at which $v(r)$ becomes infinite. Thus Theorem 1 shows that the solution of (1) with vanishing condition becomes infinite at a point in D.

REMARK 1. If for any fixed $r>0$, $h(r, v)=O(v^{1+\lambda})$, $\lambda>0$, as $v\to+\infty$, then the integral in Condition (G), v) converges.

REMARK 2. If $h(v)$ is a Lipschitz continuous function defined for all values of v such that i) $h(v)>0$ for $v\geq 0$, and ii)

$$\int_0^{\infty}\Big[\int_0^{x} h(z)\, dz\Big]^{-\frac{1}{2}} dx=+\infty,$$

then a unique solution $v(r)$ of (2) and (3) exists, as is easily verified, in the interval $(0, \infty)$. In this case, if $h(v)\geq f(x, u)$ for $x\in\overline{D}$ and $v\geq u$, we can take the solution $v(r)$ as the function $\overline{\omega}(x)$ required in the existence theorem in [1].

References

[1] S. Aizawa: On the characteristic initial value problem for non-linear wave equations in two space variables, Funkcialaj Ekvacioj, **3**, 115–146 (1961).

[2] J. B. Keller: On solutions of $\Delta u=f(u)$, Comm. Pure Appl. Math., **10**, 503–510 (1957).

[3] J. B. Keller: On solutions of nonlinear wave equations, Ibid., **10**, 523–530 (1957).