16. *Differentiability of the Generalized Solution of a Non-linear Wave Equation*

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1. In a paper [1], the author discussed the characteristic initial value problem for the non-linear wave equation

$$(1) \qquad \Box u \equiv \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = f(x_1, x_2, x_3, u)$$

in two space variables and established the existence of a generalized solution of (1) satisfying vanishing initial condition. In this note we shall show some differentiability properties of the generalized solution of (1) with vanishing condition. The main aim of this note is, however, to prove the existence of an ordinary (twice continuously differentiable) solution of (1) with vanishing condition. The proofs will be based on a comparison theorem stated in §2.

Notation.[1] The letters x, ξ, etc. will stand for the points (x_1, x_2, x_3), (ξ_1, ξ_2, ξ_3) in the space time of three dimensions, x_1 corresponding to the time variable and x_2, x_3 corresponding to the space variables. The Lorentz metric associated with (1) is defined as the form

$$(x, y) = x_1 y_1 - x_2 y_2 - x_3 y_3$$

for the scalar product of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. *All metric notions are to be interpreted according to this Lorentz metric.* The (Lorentzian) distance between two points x and ξ will always be denoted by r, while the distance of a point x from the origin will be denoted by r_x. The volume element $d\xi_1 d\xi_2 d\xi_3$ will be abbreviated to $d\xi$.

Let S be the direct characteristic cone of (1) with vertex at the origin and let Σ be a (open) space-like surface which, together with S, encloses a domain D. Denote by S_Σ that (bounded) portion of S which is cut off by Σ.

For any point x in D, denote by C^x the retrograde characteristic cone with vertex at x. Denote further by D_S^x the subdomain of D which is enclosed by C^x and S, and denote also by S^x that (bounded) portion of S which is cut off by C^x. The one dimensional intersection of C^x and S will be denoted by s^x and its line element by ds.

Let $\varphi(x)$ be in $C^1[S_\Sigma]$ and let $\underline{\omega}(x)$ and $\overline{\omega}(x) \in C[\overline{D}] \frown C^1[S_\Sigma]$[2] be such that they are expressible in $D \smile \Sigma$ in the form

1) See also [1] or [2].
2) $\overline{D}$ denotes the closure of D.

$$(2)\quad \begin{aligned} \underline{\omega}(x)&=\frac{1}{2\pi}\int_{D_S^x}\frac{\square\underline{\omega}(\xi)}{r}d\xi+\underline{\omega}(0)-\frac{1}{\pi}\int_{s^x}ds\int_0^{R_x}\frac{d\underline{\omega}}{dR}R^{-\frac{1}{2}}\,dR,^{3)}\\ \overline{\omega}(x)&=\frac{1}{2\pi}\int_{D_S^x}\frac{\square\overline{\omega}(\xi)}{r}d\xi+\overline{\omega}(0)-\frac{1}{\pi}\int_{s^x}ds\int_0^{R_x}\frac{d\overline{\omega}}{dR}R^{-\frac{1}{2}}\,dR\end{aligned}$$

respectively, where $\square\underline{\omega}(x)$ and $\square\overline{\omega}(x)$ are in $C[\overline{D}]$.

We assume that the following inequalities hold on S_Σ.

$$(3)\quad \underline{\omega}(0)\leq\varphi(0)\leq\overline{\omega}(0),\quad \frac{\partial\underline{\omega}}{\partial\lambda_x}\leq\frac{\partial\varphi}{\partial\lambda_x}\leq\frac{\partial\overline{\omega}}{\partial\lambda_x}$$

where λ_x denotes the generator of S through a point x on S_Σ and the differentiation is carried out in the direction of λ_x toward infinity.

$$\mathfrak{D}=\{(x,u);\ x\in\overline{D},\ \underline{\omega}(x)\leq u\leq\overline{\omega}(x)\}.$$

2. Let $u(x)$ be a generalized solution in D of equation (1) satisfying the initial condition $u(x)=\varphi(x)$ on S_Σ. Then, by definition,[4] $u(x)$ is a continuous solution of the non-linear integral equation

$$u(x)=\frac{1}{2\pi}\int_{D_S^x}\frac{f(\xi,u(\xi))}{r}d\xi+\varphi(0)-\frac{1}{\pi}\int_{s^x}ds\int_0^{R_x}\frac{d\varphi}{dR}R^{-\frac{1}{2}}\,dR$$

of the Volterra type.

Hence, comparing this expression with (2) and (3), we can prove the following comparison theorem.

THEOREM 1. *Let $f(x,u)$ be continuous in a domain $\Delta: x\in D$, $-\infty<u\leq\overline{\omega}(x)$ and let the inequality $f(x,u)\leq\square\overline{\omega}(x)$ hold in Δ.*

Assume that $u(x)$ is a generalized solution in D of (1) with the initial condition $u(x)=\varphi(x)$ on S_Σ and that $\overline{\omega}(0)>\varphi(0)$. Then the inequality $u(x)<\overline{\omega}(x)$ holds in $\overline{D}$.

REMARK. It is evident that a similar theorem holds for $\underline{\omega}(x)$.

3. In this paragraph we shall prove the existence of a unique ordinary solution of (1) with vanishing initial condition. In what follows, we assume that $\underline{\omega}(x)$ and $\overline{\omega}(x)$ satisfy the inequalities (3) with $\varphi(x)\equiv 0$ on S_Σ and that $\square\underline{\omega}(x)\leq 0\leq\square\overline{\omega}(x)$ in $\overline{D}$.

The following lemma is due to [1].

LEMMA 1. *If $f(x)$ is in $C[\overline{D}]$, then the function $u(x)$ defined in $D\cup\Sigma$ by the expression*

$$(4)\quad u(x)=\frac{1}{2\pi}\int_{D_S^x}\frac{f(\xi)}{r}d\xi$$

is a unique generalized solution in D of the inhomogeneous equation $\square u=f(x)$ with vanishing initial condition.

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3) $R=r^2$ and $R_x=r_x^2$. The integral is extended over S^x.

4) See Definition 1.2 and Corollary 1.2 in [1].

LEMMA 2. *If* $f(x)$ *is in* $C^{1,1}[\bar{D}]$[5] *and* $f(x)=0$ *on* S_Σ, *then the function* $u(x)$ *defined in* $D\smile\Sigma$ *by* (4) *is in* $C^1[\bar{D}]\frown C^2[D]$ *and* $\partial^2u/\partial x_i\partial x_j$ *is expressible in* D *in the form*

$$\frac{\partial^2 u}{\partial x_i\partial x_j}=\frac{1}{2\pi}\int_{S^x}\frac{1}{r}\frac{\partial f}{\partial \xi_j}\langle d\xi\rangle_i+\frac{1}{2\pi}\int_{D^x_S}\frac{1}{r}\frac{\partial^2 f}{\partial \xi_j\partial \xi_i}d\xi \quad (i,j=1,2,3)$$

where $\langle d\xi\rangle_i=d\xi_2d\xi_3$ $(i=1)$ *and* $\langle d\xi\rangle_i=+d\xi_jd\xi_k$ *or* $-d\xi_jd\xi_k$ *according as* $\xi_i<0$ *or* $\xi_i>0$ $(i\neq 1)$.

Proof. Setting

$$I^\alpha f(x)=\frac{1}{H_3(\alpha)}\int_{D^x_S}r^{\alpha-3}f(\xi)\,d\xi,$$

where $H_3(\alpha)=\pi^{\frac{1}{2}}2^{\alpha-1}\Gamma(\alpha/2)\Gamma((\alpha-1)/2)$, and integrating by parts, we have for sufficiently large α

$$(5)\qquad \begin{aligned}\frac{\partial^2 I^\alpha f(x)}{\partial x_i\partial x_j}&=\frac{1}{H_3(\alpha)}\int_{D^x_S}\frac{\partial^2 r^{\alpha-3}}{\partial x_i\partial x_j}f(\xi)\,d\xi\\&=\frac{1}{H_3(\alpha)}\int_{D^x_S}\frac{\partial^2 r^{\alpha-3}}{\partial \xi_i\partial \xi_j}f(\xi)\,d\xi=-\frac{1}{H_3(\alpha)}\int_{D^x_S}\frac{\partial r^{\alpha-3}}{\partial \xi_i}\frac{\partial f}{\partial \xi_j}d\xi\\&=\frac{1}{H_3(\alpha)}\int_{S^x}r^{\alpha-3}\frac{\partial f}{\partial \xi_j}\langle d\xi\rangle_i+\frac{1}{H_3(\alpha)}\int_{D^x_S}r^{\alpha-3}\frac{\partial^2 f}{\partial \xi_j\partial \xi_i}d\xi,\end{aligned}$$

since $r^{\alpha-3}=\partial r^{\alpha-3}/\partial\xi_i=0$ on C^x for $\alpha>5$ and, by assumption, $f(x)=0$ on S_Σ.

Hence the analytic continuation of (5) to $\alpha=2$ yields the desired expression.

DEFINITION. Let $f(x)$ be continuous in D. Then $f(x)$ is said to satisfy *Condition* (B) if the function $u(x)$ defined in $D\smile\Sigma$ by (4) is in $C^1[\bar{D}]\frown C^2[D]$ and $\|u\|^2_D<+\infty$.

REMARK. It is easily seen that $f(x)$ satisfies Condition (B) if $f(x)$ is in $C^{1,1}[\bar{D}]$ and coincides with a polynomial in x on S_Σ.

Setting $f(x,u)=g(x,u)+f(x,0)$, we now make the following

Assumptions. i) $g(x,u)$ is in $C^2[\mathfrak{D}]$ and non-decreasing with respect to u,

ii) $f(x,0)$ is in $C[\bar{D}]$ and satisfies Condition (B).

Under Assumptions i), ii), we can prove the following lemmas.

LEMMA 3. *Let* $u(x)$ *be a generalized solution of* (1) *with vanishing condition such that* $\|u\|^1_D<+\infty$. *Then the inequality*

$$\left|\frac{\partial u}{\partial x_1}\right|,\ \left|\frac{\partial u}{\partial x_2}\right|,\ \left|\frac{\partial u}{\partial x_3}\right|\le e^{\alpha x_1}+M$$

holds in D, *where* α *is a constant depending only on Assumptions*

5) $C^{1,1}[\bar{D}]$ denotes the set of functions in $C^1[\bar{D}]$ whose first derivatives satisfy a Lipschitz condition in $\bar{D}$.

i), ii) *and* D, *and* M *is a constant such that* $|\partial h/\partial x_i| \leq M$ *in* $\overline{D}$, $h(x)$ *being the function defined in* $D \smile \Sigma$ *by* (4) *with* $f(x)$ *replaced by* $f(x, 0)$.

For the proof, see Lemma 3.5 in [1].

LEMMA 4. *Let* $u(x)$ *be a solution in* $C^1[\overline{D}] \frown C^2[D]$ *of* (1) *with vanishing condition such that* $\|u\|_D^2 < +\infty$.

Then the inequality

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| \leq e^{\beta x_1} + N \quad (i, j = 1, 2, 3)$$

holds in D, *where* β *and* N *are constants depending only on Assumptions* i), ii) *and* D.

Proof. In virtue of Lemma 2, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial^2 h}{\partial x_i \partial x_j} + \frac{1}{2\pi} \int_{S^x} \frac{1}{r}\left(\frac{\partial g}{\partial \xi_j} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial \xi_j}\right) \langle d\xi \rangle_i \\ &+ \frac{1}{2\pi} \int_{D_S^x} \frac{1}{r}\left(\frac{\partial^2 g}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 g}{\partial \xi_j \partial u}\frac{\partial u}{\partial \xi_i} + \frac{\partial^2 g}{\partial \xi_i \partial u}\frac{\partial u}{\partial \xi_j} + \frac{\partial g}{\partial u}\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right) d\xi. \end{aligned} \tag{6}$$

If we set

$$v(x) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial^2 h}{\partial x_i \partial x_j} - \frac{1}{2\pi} \int_{S^x} \frac{1}{r}\left(\frac{\partial g}{\partial \xi_j} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial \xi_j}\right) \langle d\xi \rangle_i, \tag{7}$$

(6) is written as

$$\begin{aligned} v(x) = \frac{1}{2\pi} \int_{D_S^x} \frac{1}{r}&\left(\frac{\partial^2 g}{\partial \xi_i \partial \xi_j} + \frac{\partial^2 g}{\partial \xi_j \partial u}\frac{\partial u}{\partial \xi_i} + \frac{\partial^2 g}{\partial \xi_i \partial u}\frac{\partial u}{\partial \xi_j}\right. \\ &\left. + \frac{\partial g}{\partial u}\left(\frac{1}{2\pi}\int_{S^\xi} \frac{1}{r}(\cdots)\langle d\eta \rangle_i + \frac{\partial^2 h}{\partial \xi_i \partial \xi_j}\right) + \frac{\partial g}{\partial u} v(\xi)\right) d\xi. \end{aligned} \tag{8}$$

Hence it follows immediately from Assumptions i), ii) and Lemma 3 that $v(x)$ is a continuous solution in $\overline{D}$ of the integral equation (8) which vanishes on S_Σ.

Now, by assumption, we can choose β so large that the inequality

$$\begin{aligned} \beta^2 \geq \left|\frac{\partial^2 g}{\partial x_i \partial x_j}\right| &+ \left(\left|\frac{\partial^2 g}{\partial x_i \partial u}\right| + \left|\frac{\partial^2 g}{\partial x_j \partial u}\right|\right)(e^{\alpha x_1} + M) \\ &+ \frac{\partial g}{\partial u}\left(\frac{1}{2\pi}\left|\int_{S^x} \frac{1}{r}(\cdots)\langle d\xi \rangle_i\right| + \left|\frac{\partial^2 h}{\partial x_i \partial x_j}\right| + 1\right) \end{aligned}$$

holds in D. Then we can take as $\overline{\omega}(x)$ in Theorem 1 the function $e^{\beta x_1}$. Hence $v(x) \leq e^{\beta x_1}$ in $\overline{D}$. Similarly $v(x) \geq -e^{\beta x_1}$ in $\overline{D}$. It thus follows from (7) and Lemma 3 that

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| \leq e^{\beta x_1} + N \quad (i, j = 1, 2, 3)$$

where N is a constant depending only on Assumptions i), ii) and D.

Now we can prove the

THEOREM 2. *Let* $f(x, u)$ *satisfy Assumptions* i), ii) *and let the* *inequality* $\square\underline{\omega}(x)\leq f(x,u)\leq\square\overline{\omega}(x)$ *hold in* $\mathfrak{D}$.

Then there exists a unique solution in $C^1[\overline{D}]\frown C^2[D]$ *of* (1) *with vanishing initial condition.*

Proof. Let $\mathfrak{F}$ be the family of all functions $v(x)$ in $C^1[\overline{D}]$ such that $v(x)$ satisfies the inequalities

$$\begin{aligned}&\underline{\omega}(x)\leq v(x)\leq\overline{\omega}(x),\\&|\partial v(x)/\partial x_i|\leq e^{\alpha x_1}+M,\\&|\partial v(x)/\partial x_i-\partial v(x')/\partial x_i|\\&\qquad\leq(e^{\beta x_1}+N)(|x_1-x_1'|+|x_2-x_2'|+|x_3-x_3'|)\end{aligned}$$

in $\overline{D}$, where $x_1\geq x_1'$, and $v(x)=0$ on S_Σ.

Then, obviously, $\mathfrak{F}$ is not empty. It is also evident that $\mathfrak{F}$ is a compact convex set in the Banach space $C^1[\overline{D}]$.[6]

Since $v(x)$ is in $C[\overline{D}]$, it follows from Lemma 1 that there exists a unique generalized solution $u(x)$ in D of the equation $\square u=f(x,v(x))$ with vanishing condition. Thus we can define a mapping T: $u(x)=T(v(x))$. Then it is seen from Lemmas 2, 3 and 4 that $u(x)$ is in $C^1[\overline{D}]\frown C^2[D]$ and that $T(\mathfrak{F})\subset\mathfrak{F}$ for sufficiently large α, β, M and N. Continuity of T in the Banach space $C^1[\overline{D}]$ is obvious.

Hence it follows from the well-known fixed point theorem of Schauder-Tychonoff that there is a function $u(x)\in\mathfrak{F}$ such that $u(x)=T(u(x))$. Then $u(x)$ is a solution in $C^1[\overline{D}]\frown C^2[D]$ of (1) with vanishing condition.

The uniqueness follows from Corollary 2.3 in [1].

4. Let $f(x,u)$ be defined in $\mathfrak{D}$ and non-decreasing with respect to u. Then, if $f(x,u)$ is sufficiently differentiable and $\partial^{|\alpha|}f(x,0)/\partial x^\alpha=0$[7] on $S_\Sigma(|\alpha|=0,1,2,\cdots)$, we can prove, by repeating the above argument, the existence of a unique sufficiently differentiable solution $u(x)$ of (1) with vanishing condition. In fact, $\partial^{|\alpha|}u/\partial x^\alpha$ $(|\alpha|=1,2,\cdots)$ is then a solution in D of the equation of the form

$$\square v=\frac{\partial f}{\partial u}v+\sum_{1\leq|\beta|<|\alpha|}g_\beta(x,u)\frac{\partial^{|\beta|}u}{\partial x^\beta}+\frac{\partial^{|\alpha|}f}{\partial x^\alpha}$$

satisfying vanishing initial condition. Thus, in particular, we have proved the

THEOREM 3. *Let* $f(x,u)$ *be a function in* $C^\infty[\mathfrak{D}]$ *satisfying the conditions*: i) $f(x,u)$ *is non-decreasing with respect to* u, ii) $\partial^{|\alpha|}f(x,0)/\partial x^\alpha=0$ *on* $S_\Sigma(|\alpha|=0,1,2,\cdots)$. *Further let the inequality* $\square\underline{\omega}(x)\leq f(x,u)\leq\square\overline{\omega}(x)$ *hold in* $\mathfrak{D}$.

6) $C^1[\overline{D}]$ is a Banach space with the norm

$$\|u\|^1_{\overline{D}}=\max_{x\in\overline{D}}|u(x)|+\sum_{j=1}^{3}\max_{x\in\overline{D}}|\partial u/\partial x_j|\qquad(u\in C^1[\overline{D}]).$$

7) $\partial x^\alpha=\partial x^{\alpha_1}\partial x^{\alpha_2}\partial x^{\alpha_3}$ and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$.

Then there exists a unique solution in $C^{\infty}[\bar{D}]$ *of* (1) *with vanishing initial condition.*

An immediate consequence of Theorem 3 is the following

COROLLARY. *Let* $f(x,u)$ *be a function in* $C^{\infty}[\mathfrak{D}]$ *satisfying the conditions*: i) $f(x,u)$ *is non-decreasing with respect to* u, ii) $\partial^{|\alpha|+\beta}f(x,u)/\partial x^{\alpha}\partial u^{\beta}=0$ *for* $x\in S_{\Sigma}$ *and* $\underline{\omega}(x)\leq u\leq\bar{\omega}(x)$ $(|\alpha|, \beta=0,1,2,\cdots)$. *Further let the inequality* $\Box\underline{\omega}(x)\leq f(x,u)\leq\Box\bar{\omega}(x)$ *hold in* $\mathfrak{D}$ *and* $\Box\underline{\omega}(x)\leq 0\leq\Box\bar{\omega}(x)$ *in* $\bar{D}$.

Then, if $\varphi(x)$ *is a polynomial, there exists a unique solution in* $C^{\infty}[\bar{D}]$ *of* (1) *with the initial condition* $u(x)=\varphi(x)$ *on* S_{Σ}.

References

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