33. *Existence Theorems on Difference-Differential Equations*

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As an application of a fixed point theorem due to Tychonov, the author [2] has obtained a theorem concerning the existence of solutions of difference-differential equations defined on a finite interval of t such that

$$x'(t)=f(t, x(t), x(t-1))$$

under the initial conditions $x(t-1)=\varphi(t)$ $(0\leqq t<1)$ and $x(0)=x_0$, where $\varphi(t)$ is a given continuous function. In [2], he imposed on $f(t,x,y)$ only the condition of continuity of $f(t,x,y)$ in (t,x,y). For the practical problems defined on an infinite interval of t, the function $f(t,x,y)$ has so restricted a form that in the sequel we shall consider the equations, in which the function f has some stronger restrictions than those in [2].

The purpose of this paper is to obtain some results concerning the existence, stability, and boundedness of solutions of difference-differential equations by making use of Tychonov's fixed point theorem.

Recently, as an application of Tychonov's theorem, Stokes [1] has discussed the same problems as above for nonlinear differential equations. His method can also be applied for difference-differential equations.

We first prove the following

THEOREM 1. *Let $F(t,x,y)$ be continuous and nonnegative in (t,x,y) and nondecreasing in x and y for fixed t in the region R defined by $0\leqq t<\infty$ and $0\leqq x\leqq f(t)$, $0\leqq y\leqq f(t)$*[1] *where*

(i) *$f(t)$ is continuous in the interval I: $0\leqq t<\infty$ and $f(0)=\alpha$ $(\geqq 0)$;*

(ii) *$f(t)$ satisfies a difference-differential inequality*

$$f'(t)\geqq F(t, f(t), f(t-1))$$

under the condition $f(t-1)=|\varphi(t)|$ $(0\leqq t<1)$ for a given continuous function $\varphi(t)$, which has the limit $\lim_{t\to 1-0}\varphi(t)$.

Then, if we define a transformation T such that

$$Tf(t)=\alpha+\int_0^t F(s, f(s), f(s-1))\,ds,$$

1) As usual, $F(t,x,y)$ may be continuously extended to the whole region $|x|<\infty$, $|y|<\infty$. (Cf. [2].)

T has at least a fixed point, that is, there exists a solution of

$$x'(t)=F(t, x(t), x(t-1)) \tag{1}$$

on I with the initial conditions $x(t-1)=|\varphi(t)|$ $(0\leqq t<1)$ and $x(0)=\alpha$.

PROOF. It follows from (ii) that

$$\begin{aligned} Tf(t)-f(t)&=\alpha+\int_0^t F(s, f(s), f(s-1))\,ds-\left(\int_0^t f'(s)\,ds+\alpha\right)\\ &=\int_0^t (F(s, f(s), f(s-1))-f'(s))\,ds\leqq 0. \end{aligned}$$

Hence, we have $Tf(t)\leqq f(t)$. On account of the properties imposed on F, we inductively obtain the monotonicity

$$T^{n+1}f(t)\leqq T^n f(t) \quad (n=0,1,2,\cdots)$$

on I. Since $F\geqq 0$ and $\alpha\geqq 0$, it follows that $\{T^n f(t)\}_{n=0}^{\infty}$ is monotone decreasing and bounded below by 0. Furthermore, since it is proved that every $T^n f(t)$ is equicontinuous, we obtain that $\lim_{n\to\infty} T^n f(t)$ converges uniformly in the interval I and the limit is a fixed point, which is a solution of the equation

$$x(t)=\alpha+\int_0^t F(s, x(s), x(s-1))\,ds.$$

This is equivalent to (1) with the initial conditions $x(t-1)=|\varphi(t)|$ $(0\leqq t<1)$ and $x(0)=\alpha$.

Now, using the above result, we shall prove the following

THEOREM 2. *In the equation*

$$x'(t)=f(t, x(t), x(t-1)) \tag{2}$$

with the initial conditions $x(t-1)=\varphi(t)$ $(0\leqq t<1)$ and $x(0)=x_0$, we suppose that the following conditions are satisfied:

(i) *$f(t, x, y)$ is continuous in (t, x, y) for the region $R: 0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$;*

(ii) *$|f(t, x, y)|\leqq F(t, |x|, |y|)$ in R, where the function F is defined as in Theorem 1;*

(iii) *there is a function $f(t)$ as in Theorem 1.*

Then, there exists a solution of (2) with the initial conditions $x(t-1)=\varphi(t)$ $(0\leqq t<1)$ and $x(0)=x_0$, where $|x_0|\leqq\alpha$.

PROOF. Let A be a set of all functions $x(t)$ continuous on I such that $|x(t)|\leqq f(t)$, where $x(t-1)=\varphi(t)$ $(0\leqq t<1)$.

In order to apply a fixed point theorem, we define a transformation T such that

$$Tx(t)=x_0+\int_0^t f(s, x(s), x(s-1))\,ds$$

for any function in A. Then, it follows from the properties mentioned above that

$$|Tx(t)| \leqq |x_0| + \int_0^t |f(s, x(s), x(s-1))|\, ds$$
$$\leqq |x_0| + \int_0^t F(s, |x(s)|, |x(s-1)|)\, ds$$
$$\leqq \alpha + \int_0^t F(s, f(s), f(s-1))\, ds.$$

Now, it follows from (iii) that there exists a function $f(t)$ such that

$$F(t, f(t), f(t-1)) \leqq f'(t)$$

on I with the conditions $f(t-1) = |\varphi(t)|$ $(0 \leqq t < 1)$ and $f(0) = \alpha$, which shows us that the inequality $|Tx(t)| \leqq f(t)$ $(t \in I)$ remains valid. This implies $TA \subset A$.

Since it is well known that A is closed, convex, and bounded in the topology suitably chosen, it follows that there exists at least a fixed point in A, which yields an integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s), x(s-1))\, ds.$$

This is equivalent to (2) with the conditions $x(t-1) = \varphi(t)$ $(0 \leqq t < 1)$ and $x(0) = x_0$, which proves the theorem.

COROLLARY 1. *In the equation $f'(t) = F(t, f(t), f(t-1))$, where F is defined as in Theorem* 1, *suppose that there exists a solution defined on I under the initial conditions $f(t-1) = |\varphi(t)|$ $(0 \leqq t < 1)$ and $f(0) = \alpha$ $(\geqq 0)$.*

Then, there exists a solution of (2) *on I under the conditions $x(t-1) = \varphi(t)$ $(0 \leqq t < 1)$ and $x(0) = x_0$, where $|x_0| \leqq \alpha$.*

COROLLARY 2. *Let the inequality $|f(t, x, y)| \leqq \lambda(t)(M(|x|) + M(|y|))$ be satisfied, where λ and M are piecewise continuous, nonnegative, M is nondecreasing, $M(0) = 0$, and the integral*

$$\int_{r_0}^{r} \frac{d\rho}{M(\rho)}$$

is divergent as $r \to \infty$ for any $r_0 (\geqq 0)$.

Then, there exists a solution of (2) *on I under the conditions $x(t-1) = 0$ $(0 \leqq t < 1)$ and $x(0) = x_0$, where $|x_0| \leqq r_0$.*

As is observed in the proof of Theorem 2, if a solution of (1) is stable (bounded), there exists a stable (bounded) solution of (2).[2] Hence, we obtain the following

THEOREM 3. *Under the assumptions in Theorem* 2, *if $f(t)$ is stable (bounded), there exists a stable (bounded) solution of* (2).

COROLLARY 3. *Under the assumptions in Theorem* 2, *if there is a stable (bounded) solution of $f'(t) = F(t, f(t), f(t-1))$ with the conditions $f(t-1) = |\varphi(t)|$ $(0 \leqq t < 1)$ and $f(0) = \alpha$, then there exists a stable*

2) Although there are many types of stability and boundedness, any type of $x(t)$ corresponds to that of $f(t)$.

(bounded) solution of (2) *with the conditions* $x(t-1)=\varphi(t)$ $(0\leqq t<1)$ *and* $x(0)=x_0$, *where* $|x_0|\leqq\alpha$.

COROLLARY 4. *Suppose that the inequality* $|f(t,x,y)|\leqq\lambda(t)(M(|x|)+M(|y|))$ *is satisfied on* R, *where* λ *and* M *are piecewise continuous, nonnegative,* M *is nondecreasing,* $M(0)=0$, *and the integral*

$$\int_0^\infty \lambda(t)\,dt$$

is convergent, but the integral

$$\int_{r_0}^{r} \frac{d\rho}{M(\rho)}$$

diverges as $r\to\infty$ *for any* r_0 $(\geqq 0)$.

Then, there exists a bounded solution of (2) *on* I *with the conditions* $x(t-1)=0$ $(0\leqq t<1)$ *and* $x(0)=x_0$, *where* $|x_0|\leqq r_0$.

As for a perturbed equation

$$x'(t)=A(t)x(t)+B(t)x(t-1)+f(t,x(t),x(t-1)), \tag{3}$$

we consider as usual the matrix equation

$$\boldsymbol{X}'(t)=A(t)\boldsymbol{X}(t)+B(t)\boldsymbol{X}(t-1) \tag{4}$$

under the initial conditions $\boldsymbol{X}(t-1)=\boldsymbol{0}$ $(0\leqq t<1)$ and $\boldsymbol{X}(0)=E$, where $\boldsymbol{0}$ is a zero matrix and E the unit matrix.

Let $R(t)$ be a nonsingular solution of (4) under the same initial conditions as above. Then, it is easily seen that $x(t)=R(t)x_0$ is a solution of a vector equation

$$x'(t)=A(t)x(t)+B(t)x(t-1)$$

with the conditions $x(t-1)=0$ $(0\leqq t<1)$ and $x(0)=x_0$.

Let $K(t,s)$ be a matrix solution of the equations

$$\frac{\partial}{\partial t}K(t,s)=A(t)K(t,s)+B(t)K(t-1,s) \qquad (0\leqq s<t-1),$$

$$\frac{\partial}{\partial t}K(t,s)=A(t)K(t,s) \qquad (0<t-1<s<t,\ 0\leqq s<t<1),$$

$$K(t,t)=1,$$

$$K(t,s)=0\ (-1\leqq t<0).$$

Then, it is observed that

$$x(t)=R(t)x_0+\int_0^t K(t,s)w(s)\,ds$$

is a solution of

$$x'(t)=A(t)x(t)+B(t)x(t-1)+w(t)$$

with the conditions $x(t-1)=0$ $(0\leqq t<1)$ and $x(0)=x_0$.

If we define a transformation T such that

$$Tx(t)=R(t)x_0+\int_0^t K(t,s)f(s,x(s),x(s-1))\,ds,$$

we obtain, by means of the same methods as before, the following

THEOREM 4. *In the equation* (3), *we suppose that the following conditions are satisfied*:

(i) $R(t)$ *and* $K(t, s)$ *are bounded, that is,* $|R(t)| \leqq C$, $|K(t, s)| \leqq C$;

(ii) $|f(t, x, y)| \leqq F(t, |x|, |y|)$, *where* F *is defined as in Theorem* 2;

(iii) *there exists a solution* $f(t)$ *such that*

$$f'(t) \geqq CF(t, f(t), f(t-1))$$

with the conditions $f(t-1)=0$ $(0 \leqq t < 1)$ *and* $f(0)=\alpha$ $(\geqq 0)$.

Then, there exists a solution of (3) *with the conditions* $x(t-1)=0$ $(0 \leqq t < 1)$ *and* $x(0)=x_0$, *where* $C|x_0| \leqq \alpha$.

COROLLARY 5. *In the equation* $f'(t)=CF(t, f(t), f(t-1))$, *suppose that for any constant* α $(\geqq 0)$ *there exists a solution with* $f(t-1)=0$ $(0 \leqq t < 1)$ *and* $f(0)=\alpha$.

Then, there exists a solution of (3) *with* $x(t-1)=0$ $(0 \leqq t < 1)$ *and* $x(0)=x_0$, *where* $C|x_0| \leqq \alpha$.

COROLLARY 6. *If* $f(t)$ *in Theorem* 3 *is stable* (*bounded*), *the solution of* (3) *is also stable* (*bounded*).

COROLLARY 7. *If* $|f(t, x, y)| \leqq \lambda(t)(M(|x|)+M(|y|))$, *where* λ *and* M *are defined as in Corollary* 2, *there exists a solution of* (3) *on* I *with the conditions* $x(t-1)=0$ $(0 \leqq t < 1)$ *and* $x(0)=x_0$, *where* $|x_0| \leqq r_0$.

COROLLARY 8. *If* $|f(t, x, y)| \leqq \lambda(t)(M(|x|)+M(|y|))$ *in* (3), *where* λ *and* M *are defined as in Corollary* 4, *there exists a bounded solution of* (3) *on* I *with the conditions* $x(t-1)=0$ $(0 \leqq t < 1)$ *and* $x(0)=x_0$, *where* $|x_0| \leqq r_0$.

References

[1] Stokes, A.: The applications of a fixed point theorem to a variety of non-linear stability problems, Contributions to the Theory of Nonlinear Oscillations. V, Ann. Math. Studies, **45**, 173–184 (1960).

[2] Sugiyama, S.: On the existence and uniqueness theorems of difference-differential equations, Kōdai Math. Sem. Rep., **12**, 179–190 (1960).