43. *On the Behaviour of Analytic Functions on the Ideal Boundary. IV*

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Let R be a covering surface over $\underline{R}$ (with null or positive boundary). If R satisfies the following two conditions, we say that R is *almost finitely sheeted.*

Condition 1). If we take sufficiently large compact set $\underline{R}'$, $n(w) \leqq M<\infty$ in $\underline{R}-\underline{R}'$, where $n(w)$ is the number of times when w is covered by R.

Condition 2). For any point of $\underline{R}$, there exists a compact circle $C(r,p)\subset\underline{R}$ such that $C(r,p)$ is mapped onto a compact domain D_ζ in the ζ-plane by a local parameter at p and that the area of any component of $f^{-1}(C(r,p))$ (let G be a component of $f^{-1}(C(r,p))$, then G is mapped onto a connected piece over D_ζ and the area of G is taken over the ζ-plane) is finite.

If $\underline{R}$ is the w-Riemann sphere and R has finite spherical area, clearly R is almost finitely sheeted.

Lemma b). *For Beurling's theorem. Suppose a covering surface R with positive boundary and with D.S. topology over $\underline{R}$ (with null or positive boundary) with another D.S. topology. If R is almost finitely sheeted and if $\omega(F\cap G,z)$ (C.P. of $F\cap G$ relative to $R-R_0$. In this case we omit $R-R_0$)>0), $\omega(F\cap \widetilde{G},z,G')>0$, where F is a closed set in B and $\widetilde{G}$ is one domain contained in $f^{-1}(C(r_1,p))$ and G' is that of $f^{-1}(C(r_2,p))$ containing $\widetilde{G}: r_1<r_2$.*

Proof. Case 1. $\omega(F\cap G\cap CB'_n,z)>0: B'_n=f^{-1}(\underline{B}_n)$, $\underline{B}_n=E\Big[w\in R, \operatorname{dist}(z,\underline{B})\leqq\frac{1}{n}\Big]$. In this case by P.C.5 we can find a compact ζ-circle Γ_1 (Γ_1 is mapped onto a circle in the ζ-plane by a local parameter) in $\underline{R}$ such that $\omega(F\cap G\cap G_1,z)>0$, $\Gamma_1\subset\Gamma^*\subset C(r_2,p)$, where G_1 is one component of $f^{-1}(\Gamma_1)$ and Γ^* is compact set mapped onto a compact domain D_ζ by the local parameter $\zeta=\zeta(w)$. Let Γ_2 be a circle in D_ζ with the same centre as Γ_1 such that $\Gamma_1\subset\Gamma_2\subset\Gamma^*$, $\operatorname{dist}(\partial\Gamma_1,\partial\Gamma_2)>0$. Let $U(\zeta)$ be a continuous function in D_ζ such that $U(\zeta)=0$ on $D_\zeta-\Gamma_2$ $U(\zeta)$ is harmonic in $\Gamma_2-\Gamma_1$ and $U(\zeta)=1$ on Γ_1. Then $\max\left(\left|\frac{\partial U(\zeta)}{\partial\xi}\right|, \left|\frac{\partial U(\zeta)}{\partial\eta}\right|\right)\leqq M<\infty: \zeta=\xi+i\eta$. Let G_2 be one component of $f^{-1}(\Gamma_2)$ containing G_1 and consider $U(z)=U(f^{-1}(w^{-1}(\zeta)))$ in G_2. Then $D(U(z))$

$\leqq M\times$ area of G_2 (taken over the ζ-plane) $\leqq L<\infty$ and $U(z)=1$ on $G\cap G_1$, $U(z)=0$ on ∂G^*-G_2, where G^* is a component of $f^{-1}(\Gamma^*)$ containing G_2. Put $U'(z)=U(z)$ in G^* and $U'(z)=0$ in $R-G^*$. Then $U'(z)=0$ on $R-G_2$ and $U(z)=1$ on $G\cap G_1$. Further put $V(z)=\min(\omega(F_m\cap G\cap G_1,z),U'(z))$. Then $V(z)$ is continuous in $R-R_0$ and $V(z)=1$ on $F_m\cap G\cap G_1$, $V(z)=0$ on $\partial R_0+(R-R_0-G_2)$ and $D(V(z))<L+D(\omega(F_m,z)):F_m=E\left[z\in\overline{R}:\mathrm{dist}(z,F)\leqq\frac{1}{m}\right]$. Let G' be one component of $f^{-1}(C(r_2,p))$ containing G^*. Then by $G'\supset G^*\supset G_2$, $V(z)=0$ on $\partial G'+(\partial R_0\cap G')$ and $V(z)=1$ on $F_m\cap G\cap G_1$. Hence by the Dirichlet principle $D(V(z))\geqq D(\omega(F_m\cap G\cap G_1,z,G'))\geqq D(\omega(F_m\cap G\cap G_1,z))>0$ for every m. Thus by putting $\widetilde{G}=G\cap G_1$ we have $\omega(F\cap\widetilde{G},z,G')>0$.

Case 2. $\omega(F\cap G\cap B',z)=\lim_n\omega(F\cap G\cap B'_n,z)>0$. Let $\underline{R}_n$ be an exhaustion of $\underline{R}$ with compact relative boundary $\partial\underline{R}_n$. Since R_0 is compact, there exists a number n_0 such that $f(R_0)\cap(\underline{R}-\underline{R}_{n_0})=0$ and $n(w)\leqq M<\infty$ in $\underline{R}-\underline{R}_{n_0}$. By $\omega(F\cap G\cap B'z)>0$, we can find circles $C(r_1,p)\subset C(r_2,p)\subset\underline{R}-\underline{R}_{n_0}$ such that $\omega(F\cap G\cap G_1\cap B',z)>0$, where G_1 is one component of $f^{-1}(C(r_1,p))$. Now since the topology on $\underline{R}$ is D. S., there exists a sequence of domains $V_n\uparrow$ such that $\omega(CV_n\cap C(r_1,p)\cap\underline{B},w)\downarrow 0$ as $n\to\infty$ and $D(\omega(C(r_1,p)\cap V_n,w,C(r_2,p)))<L_n<\infty$. Now since $\underline{R}_{n_0}$ is compact, we have $\omega(CV_n\cap C(r_1,p)\cap B,w,\underline{R}-\underline{R}_{n_0})\downarrow 0$ by $\omega(CV_n\cap C(r_1,p)\cap\underline{B},w)\downarrow 0$ as $n\to\infty$. On the other hand, $n(w)\leqq M$ in $\underline{R}-\underline{R}_{n_0}$. Put $U_n(z)=\omega(CV_n\cap C(r_1,p)\cap\underline{B},w,\underline{R}-\underline{R}_{n_0})$ in $f^{-1}(\underline{R}-\underline{R}_{n_0})$ and $U_n(z)=0$ in $R-f^{-1}(\underline{R}-\underline{R}_{n_0})$. Then $U_n(z)=1$ on $(G_1\cap f^{-1}(CV_n)\cap B')\subset(G_1\cap f^{-1}(CV_n)\cap B)$ and $D(U_n(z))\leqq MD(\omega(CV_n\cap C(r_1,p)\cap\underline{B},w,\underline{R}-R_{n_0}))\downarrow 0$ as $n\to\infty$. Whence by the Dirichlet principle $D(\omega(F\cap G\cap G_1\cap B'\cap f^{-1}(CV_n),z))\leqq D(U_n(z))\downarrow 0$ as $n\to\infty$ and $\omega(F\cap G\cap G_1\cap B'\cap f^{-1}(CV_n),z)\downarrow 0$. By $\omega(F\cap G\cap G_1\cap B'\cap f^{-1}(V_n),z)+\omega(F\cap G\cap B'\cap f^{-1}(CV_n),z)\geqq\omega(F\cap G\cap G_1\cap B',z)$ we have

$$\omega(F\cap G\cap G_1\cap B'\cap f^{-1}(V_n),z)>0 \text{ for a number } n'.$$

Now since $D(\omega(C(r_1,p)\cap V_{n'},w,C(r_2,p)))<L_{n'}<\infty$, there exists a harmonic function $V(z)$ in G_2-G_1 such that $V(z)=1$ in $G_1\cap V_{n'}$, $V(z)=0$ on ∂G_2 and $D(V_{n'}(z))<ML'$, where G_2 is a component of $f^{-1}(C(r_2,p))$ containing G_1. Hence by the Dirichlet principle $0<D(\omega(F\cap G\cap G_1\cap f^{-1}(V_{n'})\cap B',z)\leqq D(\omega(F\cap G\cap G_1\cap f^{-1}(V_{n'})\cap B',z,G_2)\leqq D(V(z))$ and $\omega(F\cap G\cap G_1\cap f^{-1}(V_{n'}),z,G_2)>0$, whence there exists at least one component $\widetilde{V}$ of $f^{-1}(V_{n'})$ such that $\omega(F\cap G\cap G_1\cap\widetilde{V},z,G_2)>0$. Put $\widetilde{G}=G\cap G_1\cap\widetilde{V}$. Then we have the lemma.

As for potentials we proved the following facts:[1]

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1) Z. Kuramochi: On the behaviour of analytic functions on the ideal boundary. I, Proc. Japan Acad., **38**, 150-155 (1962).

1). *Let R be a Riemann surface with positive boundary. We suppose α-Martin's topology is defined on $R+B^{\alpha}$. Then $B^{\alpha}-B_1^{\alpha}=B_0^{\alpha}$ is an F_{σ} set of harmonic measure (capacity) zero, where $\alpha=K$* or *N.*

Lemma 1. a). *If $G_i \overset{\alpha}{\ni} p, i=1,2,\cdots,l$, then $\bigcap^{l} G_i \overset{\alpha}{\ni} p$.* b). *If $G \overset{\alpha}{\ni} p$, then* $\operatorname{int}(CG) \overset{\alpha}{\not\ni} p$. c). *$v_n(p) \overset{\alpha}{\ni} p$. Hence we can define $M^{\alpha}(p)=\bigcap \overline{f(G_i)}$: $G_i \overset{\alpha}{\ni} p \in B_1^{\alpha}$ and $M^{\alpha}(p)$ is one point or a continuum.*

Lemma 2. *Let G and G' be domains such that $w(F\cap B\cap G,z,G')>0(\omega(F\cap B\cap G,z,G')>0)$, then there exists at least one point $p\in F\cap B_1^{\alpha}$ such that $G \overset{\alpha}{\ni} p$.*

Lemma 3. *If* $\operatorname{dia}(C)<\delta_0$, *any component of $f^{-1}(C)$ does not contain any point $p\in B_1^{\alpha}$ α-approximately such that $\delta M(p)>\delta_0$, where δ means diameter.*

Lemma 4. *If $M^{\alpha}(p)=q$ (one point), then there exists an asymptotic path L tending to p on which $f(z)\to q$.*

Lemma 5. *Let G be a domain in R. Then $E[p\in B_1^{\alpha}: G \overset{\alpha}{\not\ni} p]$ is a G_{δ} set in B_1^{α} (since B_0^{α} is an F_{σ} set, this is also a G_{δ} set in $\overline{R}$). Let $\{C_{n,i}\}$ be a system of circles with radius $\frac{1}{n}$ such that any circle $C_{3n,j}$ with radius $\frac{1}{3n}$ is contained in a certain $C_{n,i}$. Put $T_{n,i}=E[p\in B_1^{\alpha}$: $p \overset{\alpha}{\not\in}$ any component of $f(C_{n,i})]$ and $S=E[p:\delta M^{\alpha}(p)>0]$. Then $S=\bigcup_n (\bigcap_i T_{n,i})$ is a $G_{\delta,\sigma}$ set in $\overline{R}$.*

Remark. *To prove the above assertions Lebesgue's or Fatou's theorems are not used.*

Theorem 4. a). *Let $F(z)$ be an analytic function from R into $\underline{R}$, where R is a Riemann surface with K-Martin's topology and $\underline{R}$ is a surface with positive boundary and with H.M. topology. Then $M^K(p)=\bigcap \overline{f(G_i)}: G_i \overset{K}{\ni} p \in B_1^K$, $B_0^K=B^K-B_1^K$ is an F_{σ} set of harmonic measure zero and $S^K=E[p\in B_1^K: \delta M^K(p)>0]$ is a $G_{\delta\sigma}$ set in B of harmonic measure zero.*

b). *Let R be a covering surface with positive boundary and with N-Martin's topology over $\underline{R}$ with D.S. topology (R has null or positive boundary). Suppose R is a covering surface of almost finitely sheeted. Then $M^N(p)$ is defined except an F_{σ} set of capacity zero and $E[p\in B_1^N: \delta M^N(p)>0]$ is a $G_{\delta\sigma}$ set of inner capacitg zero.*

Proof. Assume $(\bigcap_i T_{n,i})$ has a closed set $F\subset B$ of positive harmonic measure (capacity). Then $w(F,z)>0(\omega(F,z)>0)$. Let $\{C_j\}$ be circles with radius $\frac{1}{20n}$ such that $\sum_j \operatorname{int} C_j \supset \underline{R}$. Then $0<w(F,z)\leqq \sum_{j,k} w(F\cap G_{j,k},z)$, where $G_{j,k}$ is a component of $f^{-1}(C_{j,k})$. Hence there exists at least

one $G_{j,k}$ such that $w(F\cap G_{j,k}, z)>0$. Put $G=G_{j,k}$. Similarly $\omega(F\cap G, z)>0$. Let C' be a circle with radius $\frac{1}{3n}$ with the same centre as C_j and let G be one domain of $f^{-1}(C')$ containing G. Then by Lemma a) $w(F\cap G, z, G')>0(\omega(F\cap\widetilde{G}, z, G')>0$ by Lemma b), where $\widetilde{G}$ is a domain in G. Hence by Lemma 2 there exists at least one point $p\in(F\cap B^{\alpha})$ such that $G'\overset{K}{\ni}p(G'\overset{N}{\ni}p)$. On the other hand, C' is contained in a certain $C_{n,i}$ with radius $\frac{1}{n}$ and by the assumption every component of $f^{-1}(C_{n,i})$ does not contain p. This is a contradiction, because G' is contained in a component of $f^{-1}(C_{n,i})$. Thus $\bigcap_i T_{n,i}$ is a set of harmonic measure zero (of inner capacity zero). Hence by Theorem 3 (proposition) S^{α} is a set of harmonic measure zero (of inner capacity zero).

Original Fatou's theorem. *Let R be a disc $|z|<1$ and let $w=f(z)$ be a bounded analytic function. Then $f(z)$ has angular limits a.e. on $|z|=1$.*

The fact that Stoilow's topology on the w-sphere is H.S. is proved without Fatou's or Lebesgue's theorem and every point on $|z|=1$ is K-minimal. Let $p\notin S^K$. Then by Lemma 4 there exists an asymptotic path L terminating at p on which $f(z)$ converges. Hence by boundedness of $f(z), f(z)$ has angular limit a.e. on $|z|=1$. Thus we have the original Fatou's theorem without using the Lebesgue's theorem. Once Fatou's theorem is proved without Lebesgue's theorem, harmonic separativity of Green's, N and K-Martin's topologies is proved without Lebesgue's theorem. Thus theorem a) is proved without Lebesgue's theorem.

By Lemma a′) we can prove without Lebesgue's theorem similarly as a) the following

Theorem 4. a′). *Let $\underline{R}$ be a Riemann surface with null-boundary and with a topology which is homeomorphic to the original topology in $\underline{R}$. If $w=f(z)$ is of F-type, then S^K is a $G_{\delta\sigma}$ set of harmonic measure zero.*

Let R be a disc $|z|<1$ and let $\underline{R}$ be the w-Riemann sphere. Every point on $|z|=1$ is N-minimal and an open set $G\overset{N}{\ni}p\left(G=v_n(p)-\mathfrak{F}:\ v_n(p)=E\left[z: \text{dist}\,(z,p)<\frac{1}{n}\right]\right)$ if and only if the closed set $\mathfrak{F}+\widehat{\mathfrak{F}}$ ($\widehat{\mathfrak{F}}$ is the symmetric set of $\mathfrak{F}$ with respect to $|z|=1$) is so thinly distributed in $v_n(p)$ as p is an irregular point for the Dirichlet problem in $v_n(p)-\mathfrak{F}-\widehat{\mathfrak{F}}$. Suppose the spherical area $A(f(z))$ of $f(z)$ is finite. Then Theorem 4. b) is valid. $M^N(p)=q$ (one point). Then there exists a neighbourhood $v(p)$ and there exist points a, b, and c such that $f(z)\neq a$, or b or c in $v(p)$. Hence $f(z)$ has angular limit at p. Next a closed set F on $|z|=1$ is of positive capacity in our sense if and only if F is of positive logarithmic capacity. Hence we have the following

Theorem of Beurling.[2] *Let $f(z)$ be an analytic function in $|z|<1$. If the spherical area is finite, $f(z)$ has angular limits on $|z|=1$ except a set of inner capacity zero.*

Theorem 4. a) and a′) have a close relation with that of Constantinescu and Cornea.[3] They proved by completely different manner under the conditions 1). $f(z)$ is of bounded type (or extended meaning 2). $\underline{R}$ has a K-Martin's topology. Their method depends on Lebesgue's theorem and on the global property of the covering surface. On the other hand, our method depend on the local property of the covering surface. Hence our method is applicable to many topologies on $\underline{R}$.