58. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces

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Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let N be a bounded normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be its point spectrum (inclusive of the multiplicity of each eigenvalue of N); let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ be an orthonormal set determining the subspace $\mathfrak{M}$ determined by all the eigenelements of N, such that φ_ν is a normalized eigenelement of N corresponding to the eigenvalue λ_ν; let $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ be an orthonormal set determining the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}$; and let L_f be the continuous linear functional associated with an arbitrary element $f\in\mathfrak{H}$. Then $\|N\psi_\mu\|^2$, $\mu=1,2,3,\cdots$, assume the same value, which will be denoted by σ; and if we choose arbitrarily a complex constant c with absolute value $\sqrt{\sigma}$ and put $\Psi_\mu=\sum_j u_{\mu j}\psi_j$, where $u_{\mu j}=(N\psi_\mu, \psi_j)/c$ and $\sum_j$ denotes the sum for all $\psi_j\in\{\psi_\mu\}$, then the equality

$$N=\sum_\nu \lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_\mu \Psi_\mu\otimes L_{\psi_\mu}$$

holds on the domain $\mathfrak{H}$ of N, and moreover the infinite matrix (u_{ij}) associated with all the elements of $\{\psi_\mu\}$ is a unitary matrix with $|u_{jj}|\neq 1$, $j=1,2,3,\cdots$, and $\|N\|=\max(\sup_\nu|\lambda_\nu|, |c|)$ [2].

Lemma. Let $M=\max(\sup_\nu|\lambda_\nu|, |c|)$; let Γ be a rectifiable closed Jordan curve containing the closed domain $D\{\lambda:|\lambda|\leqq M\}$ inside itself; let f_α and g_α, $\alpha=1,2,3,\cdots,m$, be arbitrarily given elements in $\mathfrak{H}$; let $\varphi_\alpha(\lambda)=((\lambda I-N)^{-\alpha}f_\alpha, g_\alpha)$ and $\Phi(\lambda)=\sum_{\alpha=1}^m\varphi_\alpha(\lambda)$; and let k be an arbitrary positive integer. Then

$$F_k(z)\equiv\frac{1}{2\pi i}\int_\Gamma\Phi(\lambda)(\lambda-z)^{-k}\,d\lambda=\begin{cases}0 \text{ (for every point } z \text{ inside } \Gamma)\\ -\Phi^{(k-1)}(z)/(k-1)! \text{ (for every point } z \text{ outside } \Gamma),\end{cases}$$

where the curvilinear integration is taken in the counterclockwise direction and $0!$ and $\Phi^{(0)}(z)$ denote 1 and $\Phi(z)$ respectively.

Proof. Let $\{K(\zeta)\}$ and $\Delta(N)$ denote the complex spectral family and the continuous spectrum of N respectively. By making use of $\{K(\zeta)\}$ we can first verify without difficulty that $(\lambda I-N)^{-\alpha}$ is a bounded normal operator for any λ belonging to the resolvent set of N. Consequently the functions $\varphi_\alpha(\lambda)$ and $\Phi(\lambda)$ both are significant for every $\lambda\in\Gamma$. In addition, it is evident that $\Phi(\lambda)$ is not only continuous but also regular on Γ and that, though $\Phi(\lambda)$ has the set $\Delta(N)\cup\{\lambda_\nu\}$ of non-regular points inside Γ, the function $F_1(z)$ defined in the statement of the present lemma is regular inside and outside Γ by the continuity of $\Phi(\lambda)$ on Γ, as is well known in the function theory.

Let $f_\alpha=\sum_\nu a_\nu^{(\alpha)}\varphi_\nu+x_\alpha$, where $x_\alpha=\sum_\mu (f_\alpha, \psi_\mu)\psi_\mu$; let $g_\alpha=\sum_\nu b_\nu^{(\alpha)}\varphi_\nu+y_\alpha$, where $y_\alpha=\sum_\mu(g_\alpha,\psi_\mu)\psi_\mu$; let $P_\alpha(\lambda)=\sum_\nu a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(\lambda-\lambda_\nu)^{-\alpha}$; and let $Q_\alpha(\lambda)=\int_{\Delta(N)}(\lambda-\zeta)^{-\alpha}d\,(K(\zeta)x_\alpha, y_\alpha)$. Then, by means of the spectral integral expression of $\varphi_\alpha(\lambda)$ we obtain $\varphi_\alpha(\lambda)=P_\alpha(\lambda)+Q_\alpha(\lambda)$ and hence $\Phi(\lambda)=\sum_{\alpha=1}^m P_\alpha(\lambda)+\sum_{\alpha=1}^m Q_\alpha(\lambda)$. Moreover, by applying the inequalities $\sum_\nu |a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}| \leqq \{\sum_\nu|a_\nu^{(\alpha)}|^2\}^{\frac{1}{2}}\{\sum_\nu|\bar{b}_\nu^{(\alpha)}|^2\}^{\frac{1}{2}}<\infty$ we can readily show that the series $P_\alpha(\lambda)$ is absolutely and uniformly convergent on Γ. Hence it is found with the help of the Cauchy theorem and the calculus of residues that

$$\begin{aligned}
\int_\Gamma P_\alpha(\lambda)(\lambda-z)^{-1}d\lambda &= \sum_\nu\int_\Gamma a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(z-\lambda_\nu)^{-1}\{(\lambda-z)^{-1}(\lambda-\lambda_\nu)^{-(\alpha-1)}-(\lambda-\lambda_\nu)^{-\alpha}\}\,d\lambda \\
&\qquad\qquad (1<\alpha\leqq m)\\
&=\sum_\nu\int_\Gamma a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(z-\lambda_\nu)^{-1}(\lambda-z)^{-1}(\lambda-\lambda_\nu)^{-(\alpha-1)}\,d\lambda\\
&=\sum_\nu\int_\Gamma a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(z-\lambda_\nu)^{-2}\{(\lambda-z)^{-1}(\lambda-\lambda_\nu)^{-(\alpha-2)}-(\lambda-\lambda_\nu)^{-(\alpha-1)}\}d\lambda\\
&\;\vdots\\
&=\sum_\nu\int_\Gamma a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(z-\lambda_\nu)^{-\alpha}\{(\lambda-z)^{-1}-(\lambda-\lambda_\nu)^{-1}\}\,d\lambda\\
&=\begin{cases}0 & \text{(for every point } z \text{ inside } \Gamma)\\ -2\pi i P_\alpha(z) & \text{(for every point } z \text{ outside } \Gamma).\end{cases}
\end{aligned}$$

Moreover it is clear that the same result as above holds for $\alpha=1$. In consequence,

$$\sum_{\alpha=1}^m\int_\Gamma P_\alpha(\lambda)(\lambda-z)^{-1}\,d\lambda=\begin{cases}0 & \text{(for every point } z \text{ inside } \Gamma)\\ -2\pi i\sum_{\alpha=1}^m P_\alpha(z) & \text{(for every point } z \text{ outside } \Gamma).\end{cases}$$

On the other hand, we have

$$\begin{aligned}
\int_\Gamma Q_\alpha(\lambda)(\lambda-z)^{-1}d\lambda &=\int_\Gamma(\lambda-z)^{-1}\int_{\Delta(N)}(\lambda-\zeta)^{-\alpha}d(K(\zeta)x_\alpha, y_\alpha)\,d\lambda\\
&=\int_{\Delta(N)}\int_\Gamma(\lambda-z)^{-1}(\lambda-\zeta)^{-\alpha}d\lambda d(K(\zeta)x_\alpha, y_\alpha)
\end{aligned}$$

by considering the limit of a sequence of approximation sums of the curvilinear integral along Γ, while

$$\begin{aligned}
\int_\Gamma(\lambda-z)^{-1}(\lambda-\zeta)^{-\alpha}d\lambda&=\int_\Gamma(z-\zeta)^{-\alpha}\{(\lambda-z)^{-1}-(\lambda-\zeta)^{-1}\}\,d\lambda\\
&\qquad\qquad(1\leqq\alpha\leqq m,\ \zeta\in\Delta(N))\\
&=\begin{cases}0 & \text{(for every point } z \text{ inside } \Gamma)\\ -2\pi i(z-\zeta)^{-\alpha} & \text{(for every point } z \text{ outside } \Gamma),\end{cases}
\end{aligned}$$

as can be shown by reasoning exactly like that applied to evaluate the integral $\int_\Gamma P_\alpha(\lambda)(\lambda-z)^{-1}d\lambda$. These results permit us to assert that

$$\sum_{\alpha=1}^{m}\int_\Gamma Q_\alpha(\lambda)(\lambda-z)^{-1}d\lambda=\begin{cases} 0 & \text{(for every point } z \text{ inside } \Gamma) \\ -2\pi i\sum_{\alpha=1}^{m}Q_\alpha(z) & \text{(for every point } z \text{ outside } \Gamma). \end{cases}$$

In consequence,

$$\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda-z)^{-1}d\lambda=\begin{cases} 0 & \text{(for every point } z \text{ inside } \Gamma) \\ -\Phi(z) & \text{(for every point } z \text{ outside } \Gamma). \end{cases}$$

Since, in addition, the function $F_1(z)$ defined by the left-hand member of the final relation is regular inside and outside Γ,

$$F_1^{(k-1)}(z)=\frac{(k-1)!}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda-z)^{-k}d\lambda \quad (z\bar{\in}\Gamma)$$
$$=\begin{cases} 0 & \text{(for every point } z \text{ inside } \Gamma) \\ -\Phi^{(k-1)}(z) & \text{(for every point } z \text{ outside } \Gamma). \end{cases}$$

Thus we obtain the required relation

$$F_k(z)=\begin{cases} 0 & \text{(for every point } z \text{ inside } \Gamma) \\ -\Phi^{(k-1)}(z)/(k-1)! & \text{(for every point } z \text{ outside } \Gamma), \end{cases}$$

as we wished to prove.

Remark. Let $\{\lambda_\nu\}$ be an arbitrarily prescribed, countably infinite, and bounded set of points in the complex plane. Since, then, there exist bounded normal operators such that each of them has the set $\{\lambda_\nu\}$ as the point spectrum [1], it is seen that the lemma established above remains true even if the set of all the accumulation points of $\{\lambda_\nu\}$ is uncountable.

Definition. In the present lemma, $\sum_{\alpha=1}^{m}a_\nu^{(\alpha)}\bar{b}_\nu^{(\alpha)}(\lambda-\lambda_\nu)^{-\alpha}$ is called the principal part of $\Phi(\lambda)$ at λ_ν, and $\sum_{\alpha=1}^{m}P_\alpha(\lambda)$ and $\sum_{\alpha=1}^{m}Q_\alpha(\lambda)$ are called the first and second principal parts of $\Phi(\lambda)$ respectively. If, for a function $S(\lambda)$ defined on the domain $G\{\lambda:|\lambda|<\infty\}$, the function $R(\lambda)=S(\lambda)-\{P(\lambda)+Q(\lambda)\}$, where $P(\lambda)$ and $Q(\lambda)$ are the first and second principal parts of $S(\lambda)$ respectively, is regular on G, then $R(\lambda)$ is called the ordinary part of $S(\lambda)$, including the case $Q(\lambda)\equiv 0$.

We shall discuss about such functions as consist of these three parts.

Theorem 1. Let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed, countably infinite, and bounded set of mutually distinct points in the complex plane such that the set of all the accumulation points of it is countable or uncountable; let $S(\lambda)$ be a function regular on the domain $D\{\lambda:|\lambda|<\infty\}$ with the exception of $\{\lambda_\nu\}$ and its accumulation points such that, in the sense of the functional analysis as stated in the earlier discussion, the principal part of $S(\lambda)$ at any λ_ν is expressible in the form $\sum_{\alpha=1}^{m_\nu} c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha}$, $(m_\nu<\infty)$, where $\sum_\nu |c_\alpha^{(\nu)}|<\infty$ for every admissible value of α under the condition that $c_\alpha^{(\nu)}=0$ for $\alpha>m_\nu$; let any accumulation point of $\{\lambda_\nu\}$, not belonging to $\{\lambda_\nu\}$ itself, be purely a non-isolated essential singularity of $S(\lambda)$, that is, let $S(\lambda)$ be so defined as to have not any term with isolated essential singularity on D; let Γ be a rectifiable closed Jordan curve oriented positively such that it contains $\{\lambda_\nu\}$ and all the accumulation points of $\{\lambda_\nu\}$ inside itself; let m be the greatest value of m_ν, $\nu=1,2,3,\cdots$; let $\varphi_\alpha(\lambda)=\sum_\nu c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha}$, where $\alpha=1,2,3,\cdots,m$, and $c_\alpha^{(\nu)}=0$ for $\alpha>m_\nu$; let $\Phi(\lambda)=\sum_{\alpha=1}^{m}\varphi_\alpha(\lambda)$; and let $R(\lambda)$ be the ordinary part of $S(\lambda)$. Then, for every point z inside Γ,

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)(\lambda-z)^{-k}d\lambda=\frac{1}{2\pi i}\int_\Gamma \{S(\lambda)-\Phi(\lambda)\}(\lambda-z)^{-k}d\lambda$$
$$=R^{(k-1)}(z)/(k-1)!,\ k=1,2,3,\cdots .$$

Proof. Let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ both be incomplete orthonormal sets in $\mathfrak{H}$ such that $\{\psi_\mu\}$ determines the orthogonal complement of the subspace $\mathfrak{M}$ determined by $\{\varphi_\nu\}$; and let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$, where the matrix (u_{ij}) is an infinite unitary matrix with $|u_{jj}|\neq 1$, $j=1,2,3,\cdots$. If we now consider the operator N defined by

$$N=\sum_\nu \lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_\mu \Psi_\mu\otimes L_{\psi_\mu},$$

where c is an arbitrarily given complex constant with absolute value not exceeding $\sup_\nu|\lambda_\nu|$, then N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$ such that φ_ν is a normalized eigenelement of N corresponding to the eigenvalue λ_ν, and the spectra of N lie on the closed domain $\{\lambda:|\lambda|\leqq\sup_\nu|\lambda_\nu|\}$ [1]. If we next put

$$f_\alpha=\sum_\nu\sqrt{c_\alpha^{(\nu)}}\varphi_\nu,\quad \bar{f}_\alpha=\sum_\nu\sqrt{\bar{c}_\alpha^{(\nu)}}\varphi_\nu,$$

where $(\sqrt{c_\alpha^{(\nu)}}\varphi_\nu, \sqrt{\bar{c}_\alpha^{(\nu)}}\varphi_\nu)=c_\alpha^{(\nu)}$, then f_α and $\bar{f}_\alpha$ both belong to $\mathfrak{M}$ in accordance with the hypothesis $\sum_\nu|c_\alpha^{(\nu)}|<\infty$. On the other hand, we can find with the aid of the complex spectral family of N that the point spectrum of $(\lambda I-N)^{-\alpha}$ is given by $\{(\lambda-\lambda_\nu)^{-\alpha}\}$ and that the eigenprojector of $(\lambda I-N)^{-\alpha}$ corresponding to the eigenvalue $(\lambda-\lambda_\nu)^{-\alpha}$ is identical with that of N corresponding to the eigenvalue λ_ν. In consequence, any function $\varphi_\alpha(\lambda)$ defined in the statement of the present theorem is given by $((\lambda I-N)^{-\alpha}f_\alpha,\bar{f}_\alpha)$ and the function $\Phi(\lambda)=\sum_{\alpha=1}^{m}\varphi_\alpha(\lambda)$ is regular on Γ. Since, in addition, the principal part of $S(\lambda)$ at any λ_ν in the sense of the functional analysis coincides with that of $\Phi(\lambda)$ at the same λ_ν, the first principal part of $S(\lambda)$ is given by $\Phi(\lambda)$. Suppose now that the set of all the accumulation points of $\{\lambda_\nu\}$ is countable. Then, by the hypotheses concerning $S(\lambda)$, the second principal part of $S(\lambda)$ vanishes on D: for otherwise the set of all the accumulation points of $\{\lambda_\nu\}$ would form a set of non-zero measure, contrary to supposition. Accordingly $S(\lambda)-\Phi(\lambda)$ gives the ordinary part $R(\lambda)$ of $S(\lambda)$ on D. On the other hand, it follows from the regularity of $R(\lambda)$ on D that

$$\frac{1}{2\pi i}\int_\Gamma R(\lambda)(\lambda-z)^{-k}d\lambda=R^{(k-1)}(z)/(k-1)!,\ k=1,2,3,\cdots,$$

for every point z inside Γ. Furthermore, in the case where any $Q_\alpha(\lambda)$ in the preceding lemma vanishes, the lemma is also valid and hence applicable to the above defined function $\Phi(\lambda)$. In consequence, we obtain the relations required in the present theorem.

Suppose next that the set of all the accumulation points of $\{\lambda_\nu\}$ is uncountable. Then, by the hypotheses on $S(\lambda)$, the second principal part of $S(\lambda)$ never vanishes: for otherwise the set of all the accumulation points of $\{\lambda_\nu\}$ would become a set of measure zero, contrary to supposition. Hence $S(\lambda)-\Phi(\lambda)$ equals the sum of $R(\lambda)$ and the second principal part of $S(\lambda)$. Thus, by virtue of the application of the preceding lemma, we also obtain the required relations.

With these results, the proof of the theorem is complete.

Theorem 2. Let $\{\lambda_\nu\}$ and $\Phi(\lambda)$ be the same notations as those in Theorem 1 respectively; let Γ be a rectifiable closed Jordan curve containing the closed domain $\mathfrak{D}\{\lambda:|\lambda|\leqq\sup_\nu|\lambda_\nu|\}$ inside itself; and let N' be an arbitrary normal operator with norm not exceeding $\sup_\nu|\lambda_\nu|$. Then

$$\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda I-N')^{-k}d\lambda=\boldsymbol{O},\quad k=1,2,3,\cdots,$$

where Γ is positively oriented and $\boldsymbol{O}$ denotes the null operator.

Proof. Let $\{K'(z)\}$ denote the complex spectral family of N'. Then, by reference to the preceding lemma, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda I-N')^{-k}d\lambda&=\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)\int_{\mathfrak{D}}(\lambda-z)^{-k}dK'(z)\,d\lambda\\&=\int_{\mathfrak{D}}\left\{\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda-z)^{-k}d\lambda\right\}dK'(z)\\&=\boldsymbol{O}:\end{aligned}$$

for the z in the integrand always remains inside Γ.

Theorem 3. Let $\{\lambda_\nu\}$, $S(\lambda)$, $R(\lambda)$, and Γ be the same notations as those in Theorem 1 respectively; and let N' be a normal operator with spectra lying inside Γ. Then

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)(\lambda I-N')^{-k}d\lambda=R^{(k-1)}(N')/(k-1)!,\quad k=1,2,3,\cdots,$$

where $R^{(0)}(N')$ denotes $R(N')$.

Proof. In the same manner as above, we can easily deduce the present theorem from Theorem 1.

References

[1] S. Inoue: Functional-representations of normal operators in Hilbert spaces and their applications, Proc. Japan Acad., **37**, 614–618 (1961).

[2] ——: On the functional-representations of normal operators in Hilbert spaces, Proc. Japan Acad., **38**, 18–22 (1962).

Addition to S. Inoue: "Some Analytical Properties of the Spectra of Normal Operators in Hilbert Spaces" (Proc. Japan Acad., **37**, No. 9, 566–570 (1961)).

Page 567, line 17: Add "for appropriately chosen N_j's" between "that" and "there".