57. On Irreducible Representations of the Lorentz Group of n-th Order

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Let L_n be the Lorentz group of n-th order, i.e. the connected component of the identity element of the group of all homogeneous linear transformations in the real n-dimensional vector space which leave the quadratic form $x_1^2+x_2^2+\cdots+x_{n-1}^2-x_n^2$ invariant.

The formulas for infinitesimal operators of the irreducible representations of L_n were indicated in the paper [1]. In the present paper we classify irreducible representations of L_n and distinguish unitary ones by the results obtained in [1]. We consider also two-valued representations. Moreover it is not difficult to distinguish irreducible representations which leave Hermitian forms invariant and to investigate these Hermitian forms.

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§1. **Preliminaries.** We use same definitions and notations as in [1]. We consider the irreducible representations $\{T_g, H\}$ which are differentiable and satisfy the assumption (U). These are determined by their $(n-1)$-infinitesimal operators $A_{2,1}, A_{3,2},\cdots, A_{n-1,n-2}$ and $B=B_{n-1}$ corresponding to the one-parameter subgroups $g_{2,1}(t)$, $g_{3,2}(t),\cdots, g_{n-1,n-2}(t)$ and $g_{n-1}(t)$ respectively. The subgroups $g_{i,i-1}(t)$ $(2\leq i\leq n-1)$ generate a maximal compact subgroup U_n (rotation group in the space $x_n=0$) and the operators $A_{i,i-1}(2\leq i\leq n-1)$ determine the representation of U_n which is induced from $\{T_g, H\}$. This representation of U_n can be decomposed into irreducible components. The operator B is determined by a row of $[n/2]-1$ integers $\alpha=(n_1, n_2,\cdots, n_{[n/2]-1})$ and a complex number c.

It is easy to see that an irreducible representation of L_n is characterized by parameters $(\alpha; c)$ in the operator B and a set of irreducible representations β of U_n which is contained in the induced representation. To every generic value $(\alpha; c)$ of parameters there corresponds one irreducible representation of L_n, and in exceptional cases two or three ones. It may be of some interest to discuss this correspondence. In these arguments it is sufficient to consider only one operator B.

§2. **Classification of irreducible representations.** There are remarkable differences according to the parity of n.

I. The case when n is odd: $n=2k+1 (k=1,2,\cdots)$.

The parameter $\alpha=(n_1,n_2,\cdots,n_{k-1})$ is a row of $(k-1)$ integers satisfying

$$0\leq n_1\leq n_2\leq\cdots\leq n_{k-1}. \tag{1}$$

From the formula for the operator B ((13) and (15) in [1]), it is seen that $(\alpha;c)$ and $(\alpha;-c)$ determine the same operator B. Therefore it is sufficient to consider only those complex numbers c whose real parts are non-negative.

Irreducible representations are divided into four classes.

1) *Representations* $\mathfrak{D}_{(\alpha;c)}$, where the number c is not a half integer or is one of half integers $l_1,l_2,\cdots,l_{k-1}$.

The parameters of B corresponding to $\mathfrak{D}_{(\alpha;c)}$ are $\alpha=(n_1,n_2,\cdots,n_{k-1})$ and c. $\mathfrak{D}_{(\alpha;c)}$ contains the irreducible representations $\beta=(m_{2k-1,1}, m_{2k-1,2},\cdots,m_{2k-1,k})$ of U_n satisfying the following condition with multiplicity 1:

$$|m_{2k-1,1}|\leq n_1\leq m_{2k-1,2}\leq n_2\leq\cdots\leq m_{2k-1,k-1}\leq n_{k-1}\leq m_{2k-1,k}<+\infty. \tag{2}$$

The parameter α can be considered as the parameter of irreducible representations of the subgroup Γ_n of U_n which is generated by $g_{i,i-1}(t)$ $(2\leq i\leq n-2)$ (rotation group in the space $x_{n-1}=x_n=0$).

Then the inequality (2) means that the representation β contains the representation α (see [2]). Consequently a given representation β of U_n is contained in $\mathfrak{D}_{(\alpha;c)}$ as often as the representation α of Γ_n is contained in β (it is known that α is contained in β at most once).

2) *Finite dimensional representations* $\mathfrak{S}_\mu$, where $\mu=(n_1,n_2,\cdots,n_k)$ is a row of integers satisfying the condition

$$0\leq n_1\leq n_2\leq\cdots\leq n_k. \tag{3}$$

The corresponding parameters are $\alpha=(n_1,n_2,\cdots,n_{k-1})$ and $c=n_k+k-1/2$ (a half integer larger than $l_{k-1}: c>l_{k-1}$). $\mathfrak{S}_\mu$ contains the representations β for which

$$|m_{2k-1,1}|\leq n_1\leq m_{2k-1,2}\leq\cdots\leq n_{k-1}\leq m_{2k-1,k}\leq n_k. \tag{4}$$

3) *Representations* $\mathrm{D}^j_{(\alpha;p)}$ $(j=1,2,\cdots,k-1)$, where $n_{j-1}<n_j$ for α and p is an integer satisfying $n_{j-1}\leq p<n_j$ (put $n_0=0$ for brevity).

The corresponding parameters are $\alpha=(n_1,n_2,\cdots,n_{k-1})$ and $c=p+j-1/2$ (a half integer between l_{j-1} and $l_j: l_{j-1}<c<l_j$). $\mathrm{D}^j_{(\alpha;p)}$ contains the representations β for which

$$\begin{aligned}|m_{2k-1,1}|\leq n_1\leq m_{2k-1,2}\leq n_2\leq\cdots\leq n_{j-1}\leq m_{2k-1,j}\leq p<n_j\leq m_{2k-1,j+1}\\ \leq\cdots\leq n_{k-1}\leq m_{2k-1,k}<+\infty.\end{aligned} \tag{5}$$

4) *Representations* $\mathrm{D}^+_{(\alpha;p)}$ and $\mathrm{D}^-_{(\alpha;p)}$, where $n_1>0$ for α and p is an integer satisfying $0<p<n_1$.

The corresponding parameters are α and $c=p-1/2$ (a half integer smaller than $l_1: c<l_1$). $\mathrm{D}^+_{(\alpha;p)}$ contains β for which

$$p\leq m_{2k-1,1}\leq n_1\leq m_{2k-1,2}\leq n_2\leq\cdots\leq m_{2k-1,k}<+\infty, \tag{6}$$

and $\mathrm{D}^-_{(\alpha;p)}$ contains β for which

$$p \leq -m_{2k-1,1} \leq n_1 \leq m_{2k-1,2} \leq n_2 \leq \cdots \leq m_{2k-1,k} < +\infty. \tag{6'}$$

The representations enumerated above are all inequivalent each other except $\mathfrak{D}_{(\alpha;\,c)}$ and $\mathfrak{D}_{(\alpha;-c)}$ in the case 1). They are irreducible and satisfy the assumption (U). They exhaust the irreducible representations with infinitesimal operators of the type indicated in [1].

II. The case when n is even: $n=2k+2(k=1,2,\cdots)$.

The parmeter $\alpha=(n_1, n_2,\cdots, n_k)$ is a row of integers satisfying

$$|n_1| \leq n_2 \leq n_3 \leq \cdots \leq n_k. \tag{7}$$

From the formula for the operator B ((17) and (19) in [1]), it is clear that the parameters $(n_1, n_2,\cdots, n_k; c)$ and $(-n_1, n_2,\cdots, n_k; -c)$ determine the same operator B. Therefore it is sufficient to consider only those numbers c whose real parts are non-negative. The arguments are quite analogous with the case I.

Irreducible representations are divided into three classes.

1) *Representations* $\mathfrak{D}_{(\alpha;\,c)}$, where the number c is not equal to an integer which is equal to one of $l_1, l_2,\cdots, l_k$ or smaller than $|l_1|$.

The parameters of B corresponding to $\mathfrak{D}_{(\alpha;\,c)}$ are $\alpha=(n_1, n_2,\cdots, n_k)$ and c. It contains the representation $\beta=(m_{2k,1}, m_{2k,2},\cdots, m_{2k,k})$ of U_n for which

$$|n_1| \leq m_{2k,1} \leq n_2 \leq m_{2k,2} \leq n_3 \leq \cdots \leq n_k \leq m_{2k,k} < +\infty. \tag{8}$$

2) *Finite dimensional representations* $\mathfrak{S}_\mu$, where $\mu=(n_1, n_2,\cdots, n_{k+1})$ is a row of $(k+1)$ integers satisfying the condition

$$|n_1| \leq n_2 \leq n_3 \leq \cdots \leq n_{k+1}. \tag{9}$$

The corresponding parameters are $\alpha=(n_1, n_2,\cdots, n_k)$ and $c=n_{k+1}+k$ (an integer larger than $l_k: c>l_k$). $\mathfrak{S}_\mu$ contains β for which

$$|n_1| \leq m_{2k,1} \leq n_2 \leq m_{2k,2} \leq \cdots \leq n_k \leq m_{2k,k} \leq n_{k+1}. \tag{10}$$

3) *Representations* $\mathrm{D}^j_{(\alpha;\,p)}(j=1,2,\cdots,k-1)$, where $n_j < n_{j+1}$ for α and p is an integer satisfying $n_j \leq p < n_{j+1}$.

The corresponding parameters are $\alpha=(n_1, n_2,\cdots, n_k)$ and $c=p+j$ (an integer between l_j and $l_{j+1}: l_j < c < l_{j+1}$). $\mathrm{D}^j_{(\alpha;\,p)}$ contains β for which

$$|n_1| \leq m_{2k,1} \leq n_2 \leq \cdots \leq n_j \leq m_{2k,j} \leq p < n_{j+1} \leq m_{2k,j+1} \leq \cdots \leq n_k \leq m_{2k,k} < +\infty. \tag{11}$$

The representations enumerated above are all inequivalent except $\mathfrak{D}_{(n_1,n_2,\cdots n_k;\,c)}$ and $\mathfrak{D}_{(-n_1,n_2,\cdots,n_k;\,-c)}$ in the case 1). There hold the analogous facts mentioned at the end of the case I.

§3. Unitary representations. A representation is unitary if and only if its operator B is Hermitian.

I. $n=2k+1$. There exist five classes of irreducible unitary representations.

i) $\mathfrak{D}_{(\alpha;\,i\rho)}$, where $i=\sqrt{-1}$ and ρ is a real number.

ii) $\mathfrak{D}_{(\alpha;\,\sigma)}$, where $\mathrm{n}_{j-1}=0<n_j$ for some $j(1 \leq j \leq k-1)$ in α and $0<\sigma<j-1/2$.

iii) $\mathrm{D}^j_{(\alpha;\,0)}$, where $n_{j-1}=0<n_j$. These are the representations of

the class 3) for which $p=0$.

iv) $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$.

v) Identity representation $\mathfrak{S}_{\mu_0}$, where μ_0 is the row for which all $n_j=0$.

II. $n=2k+2$. There exist four classes of irreducible unitary representations.

i) $\mathfrak{D}_{(\alpha;i\rho)}$, where ρ is a real number.

ii) $\mathfrak{D}_{(\alpha;\sigma)}$, where $n_j=0<n_{j+1}$ for some j $(1\leq j\leq k-1)$ in α and $0<\sigma<j$.

iii) $D^j_{(\alpha;0)}$, where $n_j=0<n_{j+1}$. These are the representations of the class 3) for which $p=0$.

iv) Identity representation $\mathfrak{S}_{\mu_0}$.

For $n=3$, and 4, some of the classes listed in §2 and §3 do not appear. For $n=5$, the situation become general and all classes really exist.

§4. **Two-valued representations.** If we consider the representations of groups locally isomorphic with L_n, there appear two-valued representations of L_n. The formulas for the operators $A_{2,1}, A_{3,2},\cdots, A_{n-1,n-2}$ and B in [1] are valid for two-valued representations, but the numbers m_{ij} and n_j are all half integers.

We mention briefly the classification of two-valued representations. The arguments are quite analogous in the case of single-valued representations and the description in §2 is valid without changes of notations and inequalities if m_{ij} and n_j are substituted by half integers.

We describe the results more exactly.

I. When n is odd: $n=2k+1$. The representations are divided into four classes as follows.

1′) Representations $\mathfrak{D}_{(\alpha;c)}$, where the number c is not equal to an integer or is one of integers $l_1, l_2,\cdots, l_{k-1}$.

The corresponding parameters are α and c. $\mathfrak{D}_{(\alpha;c)}$ contains the representations β which satisfy the inequality similar to (2).

2′) Finite dimensional representations $\mathfrak{S}_\mu$.

3′) Representations $D^j_{(\alpha;p)}(j=1,2,\cdots,k-1)$, where p is a half integer satisfying $n_{j-1}\leq p<n_j$.

4′) Representations $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$.

II. When n is even: $n=2k+2$, the representations are divided into three classes.

1′) $\mathfrak{D}_{(\alpha;c)}$; 2′) $\mathfrak{S}_\mu$; 3′) $D^j_{(\alpha;p)}$ $(j=1,2,\cdots,k-1)$.
Here l_j is a half integer and p is an integer.

If we consider irreducible unitary representations, some differences are found as for the results in §3.

I. $n=2k+1$. There exist only two classes of irreducible unitary representations.

i′) Representations $\mathfrak{D}_{(\alpha;\, i\rho)}$, where $i=\sqrt{-1}$ and ρ is real number.

iv′) Representations $D^{+}_{(\alpha;\, p)}$ and $D^{-}_{(\alpha;\, p)}$.

II. $n=2k+2$. In this case there exists only one class.

i′) Representations $\mathfrak{D}_{(\alpha;\, i\rho)}$, where ρ is a real number.

We shall discuss explicite construction of these representations on another occasion (for the case $n=5$, see [3]).

References

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