52. 2-Primary Components of the Homotopy Groups of Some Lie Groups

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This is a preliminary report of results concerning the generators of 2-primary components of the homotopy groups of $SO(n)$, $SU(n)$ and $Sp(n)$. The proofs will be given elsewhere.

1. Let $R_n (n \geqq 2)$ denote the special orthogonal group $SO(n)$, $U_n (n \geqq 1)$ the special unitary group $SU(n)$, and $Sp_n (n \geqq 1)$ the symplectic group $Sp(n)$.

Let $i^{m,n}: R_n \to R_m$, $i'^{m,n}: U_n \to U_m$, $i''^{m,n}: Sp_n \to Sp_m$, $(n \leqq m)$
$l^{2n}: Sp_n \to U_{2n}$, $k^{2n}: U_n \to R_{2n}$, $(n \geqq 1)$,
be the inclusion maps.

Let us denote the projections and the characteristic classes of the bundles R_{n+1}, U_{n+1}, and Sp_{n+1}, by
$$p: R_{n+1} \to S^n, \quad p': U_{n+1} \to S^{2n+1}, \quad p'': Sp_{n+1} \to S^{4n+3},$$
$$T_n \in \pi_{n-1}(R_n), \quad T'_n \in \pi_{2n}(U_n), \quad T''_n \in \pi_{4n+2}(Sp_n),$$
respectively.

Let Z, Q, and C denote the field of complex numbers, algebras of quaternions, and of Cayley numbers over the field of real numbers, respectively.

Then the spheres S^1, S^2, S^3, S^6, and S^7 are represented as follows:
$$S^1 = \{z \in Z;\ z\bar{z} = 1\}, \quad S^3 = \{q \in Q;\ q\bar{q} = 1\}, \quad S^7 = \{c \in C;\ c\bar{c} = 1\},$$
$$S^2\{q' = x_1 k + x_2 j + x_3 i + x_4 \in S^3;\ x_4 = 0\},$$
$$S^6 = \{c = (q, q') \in S^7;\ x_8 = 0\}, \quad \text{where } q' = x_5 k + x_6 j + x_7 i + x_8.$$
Define the maps
$$\sigma'^1_1: S^1 \to U_1, \quad \sigma^2_1: S^1 \to R_2, \quad \sigma''^1_3: S^3 \to Sp_1, \quad \sigma'^2_3: S^3 \to U_2,$$
$$\sigma^4_3: S^3 \to R_4, \quad \sigma^8_7: S^7 \to R_8, \quad \rho^3_3: S^3 \to R_3, \quad \rho^7_7: S^7 \to B_7,$$
as follows:
$$\sigma'^1_1(z)(z') = zz', \ (z, z' \in S^1), \quad \sigma^2_1 = k^2 \circ \sigma'^1_1,$$
$$\sigma''^1_3(q)(q') = qq' \ (q, q' \in S^3), \quad \sigma'^2_3 = l^2 \circ \sigma''^1_3, \quad \sigma^4_3 = k^4 \circ \sigma'^2_3,$$
$$\sigma^8_7(c)(c') = cc' \ (c, c' \in S^7),$$
$$\rho^3_3(q)(q') = qq'\bar{q}, \ (q \in S^3,\ q' \in S^2),$$
$$\rho^7_7(c)(c') = cc'\bar{c}, \ (c \in S^7,\ c' \in S^6).$$
We denote e.g. $i^{m,n} \circ \sigma^n_p$ by σ^m_p.

We denote by q and q' the well known isomorphisms:
$$q: \pi_m(Sp_2) \to \pi_m(R_5), \quad q': \pi_m(U_4) \to \pi_m(R_6), \ (m \geqq 2).$$
Most of generators of the homotopy groups of R_n, U_n and Sp_n, will be represented in terms of the elements defined above and of

those of the homotopy groups defined in our former paper [6]. If a generator can not be represented in this way, we use the notation $r^n_m \in \pi_m(R_n)$ etc. We also denote $i^{m,n} r^n_p$ by r^m_p.

The first results on homotopy groups of R_n, U_n, and Sp_n were announced by H. Toda [1]. R. Bott [2] proved the periodicity of the stable homotopy groups of R_n, U_n and Sp_n, and Toda [3], Kervaire [4], and Matsunaga [5], calculated some unstable homotopy groups of U_n. Using these results, the exact sequence of the bundles $R_{n+1} \to S^n$, $U_{n+1} \to S^{2n+1}$, and $Sp_{n+1} \to S^{4n+3}$, and the diagram (1.2) of [6], we can obtain the following results.

2. In the following sections, we always consider 2-primary components of groups. For simplicity, we shall denote e.g. $\pi_m(Sp_n)$ to mean the 2-primary components of $\pi_m(Sp_n)$; and use the terms such as "equal", "isomorphic", in the sense of C_2 [7].

Non-zero generators of $\pi_m(Sp_n)$ $(m \leqq 13)$ are listed as follows:

(2.1) $\sigma''^n_3 \in \pi_3(Sp_n)$ $(n \geqq 1)$, $\sigma''^n_3 \circ \eta_3 \in \pi_4(Sp_n)$ $(n \geqq 1)$,
$\sigma''^n_3 \circ \eta_3 \circ \eta_4 \in \pi_5(Sp_n)$ $(n \geqq 1)$, $\sigma''^1_3 \circ \alpha_3 \in \pi_6(Sp_1)$,
$\sigma''^1_3 \circ \eta_3 \circ \nu_4 \in \pi_7(Sp_1)$, $s^n_7 \in \pi_7(Sp_n)$ $(n \geqq 2)$,
$\sigma''^1_3 \circ \eta_3 \circ \nu_4 \circ \eta_7 \in \pi_8(Sp_1)$, $i''^{n,2} \circ T''_2 \in \pi_{10}(Sp_n)$ $(n=2,3)$,
$\sigma''^n_3 \circ \varepsilon_3$ $(n=1,2)$, s^n_{11} $(n \geqq 3)$, $\in \pi_{11}(Sp_n)$,
$\sigma''^n_3 \circ \eta_3 \circ \varepsilon_4$ $(n=1,2)$, $\sigma''^n_3 \circ \delta_3$ $(n \geqq 1)$, $\in \pi_{12}(Sp_n)$,
$\sigma''^n_3 \circ \eta_3 \circ \delta_4 \in \pi_{13}(Sp_n)$ $(n \geqq 1)$, $\sigma''^1_3 \circ \varepsilon'_3 \in \pi_{13}(Sp_1)$,
$T''_2 \circ \nu_{10} \in \pi_{13}(Sp_2)$,

where the orders of s^2_7 and s^3_{11} are both ∞.

Then we have the following relations:

(2.2) $p''_*(s^2_7) = 12\iota_7$, $p''_*(T''_2) = \nu_7$, $p''_*(s^3_{11}) = 5!\iota_{11}$,
$T''_1 = \sigma''^1_3 \circ \alpha_3$, $s^2_7 \circ \nu_7 = 4T''_2$, $\sigma''^2_3 \circ \varepsilon_3 = T''_2 \circ \eta_{10}$,
$\sigma''^2_3 \circ \varepsilon'_3 = 2T''_2 \circ \nu_{10}$.

3. Non-zero generators of $\pi_m(U_n)$ $(m \leqq 13)$ are listed as follows:

(3.1) $\sigma'^n_1 \in \pi_1(U_n)$ $(n \geqq 1)$, $\sigma'^n_3 \in \pi_3(U_n)$ $(n \geqq 2)$,
$\sigma'^2_3 \circ \eta_3 \in \pi_4(U_2)$, $\sigma'^2_3 \circ \eta_3 \circ \eta_4 \in \pi_5(U_2)$, $u^n_5 \in \pi_5(U_n)$ $(n \geqq 5)$
$\sigma'^n_3 \circ \alpha_3 \in \pi_6(U_n)$ $(n=2,3)$, $\sigma'^2_3 \circ \eta_3 \circ \nu_4 \in \pi_7(U_2)$,
$u^n_7 \in \pi_7(U_n)$ $(n \geqq 4)$, $\sigma'^2_3 \circ \eta_3 \circ \nu_4 \circ \eta_7 \in \pi_8(U_2)$,
$u^3_5 \circ \nu_5 \in \pi_8(U_3)$, $T'_4 \in \pi_8(U_4)$, $T'_4 \circ \eta_8 \in \pi_9(U_4)$,
$u^n_9 \in \pi_9(U_n)$ $(n \geqq 5)$, u^n_{10} $(n=3,4)$, $i'^{n,4} \circ l^4 \circ T''_2$ $(n=4,5)$, $\in \pi_{10}(U_n)$,
$\sigma'^2_3 \circ \varepsilon_3 \in \pi_{11}(U_2)$, u^n_{11} $(n=3,4)$, $i'^{n,6} \circ l^6 \circ s^3_{11}$ $(n \geqq 6)$, $\in \pi_{11}(U_n)$,
$\sigma'^2_3 \circ \eta_3 \circ \varepsilon_4$, $\sigma'^2_3 \circ \delta_3$, $\in \pi_{12}(U_2)$, $u'^n_{12} \in \pi_{12}(U_n)$ $(n=3,4)$,
$u^5_{12} \in \pi_{12}(U_5)$, $T'_6 \in \pi_{12}(U_6)$, $\sigma'^2_3 \circ \eta_3 \circ \delta_4 \in \pi_{13}(U_2)$,
$\sigma'^n_3 \circ \varepsilon'_3$ $(n=2,3)$, $i'^{n,4} \circ l^4 \circ T''_2 \circ \nu_{10}$, $\in \pi_{13}(U_n)$, $T'_6 \circ \eta_{12} \in \pi_{13}(U_6)$,
$u^n_{13} \in \pi_{13}(U_n)$ $(n \geqq 7)$,

where the orders of the elements u^3_5, u^4_7, u^5_9, u^3_{10}, u^3_{11}, u'^3_{12}, u^5_{12}, and u^7_{13} are ∞, ∞, ∞, 2,4,4,8, and ∞, respectively. I do not know whether they can be represented in terms of the other known elements.

We have the following relations;

(3.2) $p'_*(u_5^3)=2\iota_5,\quad p'_*(u_7^4)=6\iota_7\quad p'_*(u_9^5)=24\iota_9,$
$p'_*(u_{10}^3)=\nu_5\circ\eta_8\circ\eta_9,\quad p'_*(u_{11}^3)=\nu_5\circ\nu_8,\quad p'_*(u'^3_{12})=\beta''_5,$
$p'_*(u_{12}^5)=4\nu_9,\quad p'_*(u_{13}^7)=6!\iota_{13}$

(3.3) $T'_2=\sigma'^2_3\circ\eta_3,\quad k^6\circ u_5^3=T_6,\quad T'_3=\sigma'^3_3\circ\alpha_3,\quad l^4\circ s_7^2=2u_7^4,$
$u_8^4=2T'_4,\quad T'_4\circ\eta_8\circ\eta_9=u_{10}^4+4l^4\circ T''_2,$
$u_{10}^5=4i'^{5,4}l^4\circ T''_2=4T'_5,\quad \sigma'^3_3\circ\varepsilon_3=2u_{11}^3,\quad u_{11}^4=T'_4\circ\nu_8,$
$\sigma'^3_3\circ\delta_3=2u'^3_{12},\quad u'^5_{12}=2u_{12}^5,\quad u_{12}^6=2T'_6,$
$\sigma'^4_3\circ\varepsilon'_3=2l^4\circ T''_2\circ\nu_{10}.$

4. Non-zero generators of $\pi_m(R_n)$ $(m\leqq 8)$ are listed as follows:

(4.1) $\sigma_1^n\in\pi_1(R_n)\ (n\geqq 2),$
$\rho_3^n\ (n=3,4),\quad \sigma_3^n\ (n\geqq 4),\ \in\pi_3(R_n),$
$\rho_3^n\circ\eta_3\ (n=3,4),\quad \sigma_3^n\circ\eta_3\ (n=4,5),\ \in\pi_4(R_n),$
$\rho_3^n\circ\eta_3\circ\eta_4\ (n=3,4),\quad \sigma_3^n\circ\eta_3\circ\eta_4\ (n=4,5),\ \in\pi_5(R_n),$
$T_6\in\pi_5(R_6),\quad \rho_3^n\circ\alpha_3\in\pi_6(R_n)\ (n=3,4),\quad \sigma_3^4\circ\alpha_3\in\pi_6(R_4),$
$\rho_3^n\circ\eta_3\circ\nu_4\ (n=3,4),\quad \rho_7^n\ (n=7,8),\quad \sigma_8^n\ (n\geqq 8),\ \in\pi_7(R_n),$
$q(s_7^2)\in\pi_7(R_5),\quad q'(u_7^4)\in\pi_7(R_6),$
$\rho_3^n\circ\eta_3\circ\nu_4\circ\nu_7\in\pi_8(R_n)\ (n=3,4),\quad \sigma_3^4\circ\eta_3\circ\nu_4\circ\eta_7\in\pi_8(R_4),$
$i^{n,6}\circ q'(T'_4)\ (n\geqq 6),\quad \rho_7^n\circ\eta_7\ (n=7,8),\quad \sigma_7^n\circ\eta_7\ (n\geqq 8),\ \in\pi_8(R_n).$

We denote the elements $q(s_7^2)$, $q'(u_7^4)$, and $q'(T'_4)$ by r'^5_7, r_7^6, and r_8^6, respectively. Then we have the following relations:

(4.2) $p_*(\sigma_1^2)=\iota_1,\quad p_*(\rho_3^3)=\eta_2,\quad p_*(\sigma_3^4)=\iota_3,$
$p_*(r'^5_7)=12\nu_4,\quad p_*(r_7^6)=\eta_5\circ\eta_6,\quad p_*(\rho_7^7)=\eta_6,$
$p_*(\sigma_7^8)=\iota_7,\quad p_*(r_8^6)=\nu_5.$

(4.3) $T_2=2\sigma_1^2,\quad T_3=0,\quad T_4=\rho_3^4+2\sigma_3^4,\quad T_7=0,$
$T_8=\rho_7^8+2\sigma_7^8,\quad T_9=r_8^9+\sigma_7^9\circ\eta_7,$

(4.4) $\rho_3^5=2\sigma_3^5,\quad r'^6_7=2r_7^6,\quad r_7^7=2\rho_7^7,\quad \rho_7^9=2\sigma_7^9,$
$r_8^{10}=\sigma_7^{10}\cdot\eta_7.$

Non-zero generators of $\pi_m(R_n)$ $(9\leqq m\leqq 13)$ are listed as follows:

(4.5) $r_8^n\circ\eta_8\ (n\geqq 6),\quad \rho_7^n\circ\eta_7\circ\eta_8\ (n=7,8),$
$\sigma_7^n\circ\eta_7\circ\eta_8\ (n\geqq 8),\ \in\pi_9(R_n);\quad T_{10}\in\pi_9(R_{10}),$
$i^{n,5}\circ q(T''_2)\ (n\geqq 5),\quad \sigma_7^n\circ\nu_7\ (8\leqq n\leqq 11),\ \in\pi_{10}(R_n);$
$k^6\circ u_{10}^3\in\pi_{10}(R_6),$
$\rho_3^n\circ\varepsilon_3\ (n=3,4),\quad \sigma_3^n\circ\varepsilon_3\ (n=4,5),\quad r_8^n\circ\nu_8\ (6\leqq n\leqq 10),$
$r_{11}^n\ (n\geqq 7),\ \in\pi_{11}(R_n);\quad T_{12}\in\pi_{11}(R_{12}),$
$\rho_3^n\circ\eta_3\circ\varepsilon_4,\quad \rho_3^n\circ\delta_3,\ (n=3,4),\quad \sigma_3^n\circ\eta_3\circ\varepsilon_4,\quad \sigma_3^n\circ\delta_3\ (n=4,5),$
$r_{12}^n\ (n=11,12),\quad i^{n,12}\circ k^{12}\circ T'_6\ (n=12,13),\ \in\pi_{12}(R_n);$
$k^6u'^3_{12}\in\pi_{12}(R_6),\quad T_{10}\circ\nu_9\in\pi_{12}(R_{10}),$
$\rho_3^n\circ\eta_3\circ\delta_4,\quad \rho_3^n\circ\varepsilon'_3,\ (n=3,4),\quad \sigma_3^n\circ\eta_3\circ\delta_4\ (n=4,5),$
$i^{n,5}q(T''_2\circ\nu_{10})\ (5\leqq n\leqq 8),\quad \sigma_7^n\circ\nu_7\circ\nu_{10}\ (8\leqq n\leqq 11),$
$r_{12}^n\circ\eta_{12}\ (n=11,12),\quad i^{n,12}\circ k^{12}\circ T'_6\circ\eta_{12}\ (n=12,13),\ \in\pi_{12}(R_n);$
$\sigma_3^4\circ\varepsilon'_3\in\pi_{13}(R_4),\quad T_{14}\in\pi_{13}(R_{14}),$

where the orders of r_{11}^7 and r_{12}^{11} are ∞ and 2, respectively.

We denote $q(T_2'')$ by r_{10}^5. Then we have the following relations:

(4.6) $p_*(r_{10}^5)=\nu_4\circ\nu_7,\quad p_*(r_{11}^7)=2[\iota_6,\iota_6],\quad p_*(r_{12}^{11})=\eta_{10}\circ\eta_{11}.$

(4.7) $T_{11}=\sigma_7^{11}\circ\nu_7.$

(4.8) $r_8^{10}\circ\eta_8=\rho_7^{10}\circ\eta_7\circ\eta_8,\quad i^{7,6}k^6u_{10}^3=4r_{10}^7,\quad r_{10}^9=2\sigma_7^9\circ\nu_7,$
$k^8\circ l^4\circ T_2''=\sigma_7^8\circ\nu_7,\quad \sigma_3^6\circ\varepsilon_3=2r_8^6\circ\nu_8,\quad \sigma_3^6\circ\delta_3=2k^6\circ u_{12}^3,$
$\sigma_3^5\circ\varepsilon_3'=2r_{10}^5\circ\nu_{10}.$

Let J denote the Whitehead J-homomorphism [6], then we have:

(4.9) $J(\sigma_1^2)=\eta_2,\quad J(\rho_3^3)=\alpha_3,\quad J(\alpha_3^4)=\nu_4,\quad J(r_7'^5)=\beta_5'',$
$J(r_7^6)=\beta_6',\quad J(\rho_7^7)=\beta_7,\quad J(\sigma_7^8)=\mu_8,\quad J(r_8^6)=\nu_6',$
$J(r_{10}^5)=\nu_5\circ\mu_8,\quad J(r_{11}^7)=0,\quad J(k^6\circ u_{12}'^3)=4[\iota_6,\iota_6]\circ\mu_{11},$
$J(r_{12}^{11})=\lambda_{11},\quad J(k^{12}\circ T_6')=\tau_{12}.$

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