76. On Distribulions and Spaces of Sequences. II

On the Multiplications of Improper Functions in Quantum Field Theory

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1. **Introduction.** In quantum field theory, the multiplication of distributions such as $\delta\cdot\delta$, $pfx^{-1}\cdot\delta$, $\delta'\cdot\delta$ etc., plays an important role. [**5**][**6**].

But the multiplications of these types can not be defined in proper sense of distribution theory. So, it will be desirable attempt to extend the notion of proper distribution, define extended multiplication and calculate these products. In [8] we introduced the spaces of sequences and generalized distributions and studied the relations between proper distributions and generalized distributions.

The multiplication in spaces of sequences is associative and commutative, which is convenient to the calculation of S matrix in quantum field theory in contrast to the other definitions such as in [4][**5**][**6**].

The weak point of the multiplication in spaces of generalized distribution is the indeterminateness of the product. But this defect is not disadvantage for the calculation of S matrix. E. Stueckelberg, A. Petermann and W. Güttinger associated these indeterminateness with the so-called "ambiguities of current field theory" and introduced renormalization constants.

In this paper we study mainly some concrete examples which is important in quantum field theory such as $\delta\cdot\delta$, $\delta\cdot x^{-1}$ etc. and show also that the Güttinger's product is contained as a special case of our multiplication.

2. **Notations and Definitions** [8]. Let $\boldsymbol{Q}$ denote the set of all sequences $\{\varphi_n\}$ of functions $\varphi_n\in\mathcal{E}$, i.e. $\boldsymbol{Q}=\{\{\varphi_n\};\ \varphi_n\in\mathcal{E}\}$. Let $\widetilde{\boldsymbol{Q}}_\tau$ denote the set of all convergent sequences in τ topology. Let $\mathfrak{O}_\tau$ denote the set of all sequences which converge to zero in τ topology. Let Q_τ denote the set of classes such that $Q_\tau\equiv\boldsymbol{Q}/\mathfrak{o}_\tau=\{\mathfrak{c}(\ |\tau|\ \widetilde{T}_\alpha),\mathfrak{c}(\ |\tau|\infty_\beta)\}$. Let $\widetilde{Q}_\tau$ be the set of all convergent classes, i.e. $\widetilde{Q}_\tau\equiv\widetilde{\boldsymbol{Q}}_\tau/\mathfrak{o}_\tau=\{\mathfrak{c}(\ |\tau|\ \widetilde{T}_\alpha)\}$.

Let $\boldsymbol{Q}^{D'}$ be the set of all convergent (in D' topology) sequences $\{\varphi_n\}$, $\varphi_n\in\mathcal{E}$. Let $Q_\tau^{D'}$ be the set of all classes $Q_\tau^{D'}\equiv\boldsymbol{Q}^{D'}/\mathfrak{o}_\tau=\{\mathfrak{c}(T|\tau|\ \widetilde{T}_\alpha),\mathfrak{c}(T|\tau|\infty_\beta)\}$, where $\varphi_n\in\mathfrak{c}(T|\tau|\ \widetilde{T}_\alpha)$ means $\varphi_n\to T$ in $\mathfrak{D}'$, $\varphi_n\to\widetilde{T}_\alpha$ in τ.

Let $\widetilde{\boldsymbol{Q}}_{\tau}^{D'}$ denote the set of all sequences which are convergent in τ and in $\tau_{(D')}$. Let $Q_{\tau}^{D'}$ denote the set of all convergent classes i.e. $\widetilde{Q}_{\tau}^{D'}\equiv\widetilde{\boldsymbol{Q}}_{\tau}^{D'}/\mathfrak{D}_{\tau}=\{\mathfrak{c}(T\mid\tau\mid\widetilde{T}_{\alpha})\}$.

Let P_{τ} be the natural mapping from $\boldsymbol{Q}$ to Q_{τ}. Let $\mathfrak{c}(T|\tau_1|\ )$ be the element of $Q_{\tau_1}^{D'}$, and let $\mathfrak{c}(S|\tau_2|\ )$ be the element of $Q_{\tau_1}^{D'}$.

Definition 1.

$\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )=\{\{\psi_n\};\psi_n=\varphi_n^T\cdot\varphi_n^S,\{\varphi_n^T\}\in\mathfrak{c}(T|\tau_1|\ ),\{\varphi_n^S\}\in\mathfrak{c}(S|\tau_2|\ )\}$,

$[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}=P_{\tau_3}\{\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )\}$.

Definition 2.

$\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )=\{\{\psi_m\};\psi_m=\varphi_{n_1(m)}^T\cdot\varphi_{n_2(m)}^S,\{\varphi_n^T\}\in\mathfrak{c}(T|\tau_1|\ ),\{\varphi_n^S\}\in\mathfrak{c}(S|\tau_2|\ )\}$,
where m, $n_1(m)$ and $n_2(m)$ are natural numbers.

$[\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}=P_{\tau_3}\{\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )\}$.

Lemma 1. $\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )\subseteq\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )$,

$[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}\subseteq[\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}$.

We call $[\mathfrak{c}(T\mid\tau_1\mid)\cdot\mathfrak{c}(S\mid\tau_2\mid)]_{(D')}$ **range of product in $(\boldsymbol{D}')$**. This definition is the extension of the definition by L. Schwartz. He defines the product for $\alpha\in(\mathcal{E})$ and $T\in(\boldsymbol{D}')$. In his definition, α is an equivalent class in the discrete topology in $(\mathcal{E})$. T is an equivalent class in (D').

3. Ranges of product. At the first step we give the range of product $[\mathfrak{c}(\delta|\tau|\widetilde{\delta})\cdot\mathfrak{c}(\delta|\tau|\widetilde{\delta})]_{(D')}$ for topology τ such that $\{\varphi_n^\delta\}\in\mathfrak{c}(\delta\mid\tau\mid\widetilde{\delta})$ satisfy the following condition; there exists a sequence of positive numbers $\{\varepsilon_n\}$ such that $\lim_{n\to\infty}\varepsilon_n=0$ and $|\varphi_n^\delta|<\varepsilon_n$ for $|x|\geqq\varepsilon_n$. (We can construct this sort of topology τ by the following neighbourhoods: $V_\varepsilon(0)=\{\varphi\mid\sup|e^{-1/x^2}\varphi|<\varepsilon\}$, [10].

Here we consider specially the meaning of elements in $[\mathfrak{c}(\delta\mid\tau\mid\widetilde{\delta})\cdot\mathfrak{c}(\delta\mid\tau\mid\widetilde{\delta})]_{(D')}$. If $\{\varphi_n^\delta\cdot\widetilde{\varphi}_n^\delta\}$ defines a distribution, then its carrier is the original point, where $\{\varphi_n^\delta\}$, $\{\widetilde{\varphi}_n^\delta\}$, $\{\psi_n^\delta\}$, $\{\widetilde{\psi}_n^\delta\}\in\mathfrak{c}(\delta|\tau|\ )$. So it must take the form $\sum_{k=0}^{N}C_k\delta^{(k)}$. Conversely selecting suitable sequence, we can construct $\sum_{k=0}^{N}C_k\delta^{(k)}$ for arbitrary C_k and N.

For example, let $\{\varphi_n^\delta\}$, $\{\widetilde{\varphi}_n^\delta\}$ be the functions shown in the figures. Then $\{\varphi_n^\delta\cdot\widetilde{\varphi}_n^\delta\}$ converges to $81\delta''/4$.

Further we can even construct $\sum_{k=0}^{\infty}C_k\delta^{(k)}$, where $\sum_{k=0}^{\infty}C_k\delta^{(k)}$ is a class $\mathfrak{c}(\ |\tau_{D'}|\infty_\beta)$ which contains the sequence $\{\varphi_n\}$ such that $\langle\{\varphi_n\}x^m\rangle\equiv\lim_n\langle\varphi_n,x^m\rangle=(-1)^mC_m\cdot m!$

From these results and others [10], we obtain the following

Theorem 1. *If the set of the sequences $\mathfrak{c}(\delta\mid\tau\mid\widetilde{\delta}')$ has the following property; there exists a sequence of positive numbers $\{\varepsilon_n\}$ such that $\lim_{n\to\infty}\varepsilon_n=0$ and $\varphi_n^\delta<\varepsilon_n$ for $x\geqq\varepsilon_n$ and $x\leqq-\varepsilon_n$, then for $\{\varphi_n^\delta\}$, $\{\widetilde{\varphi}_n^\delta\}\in$*

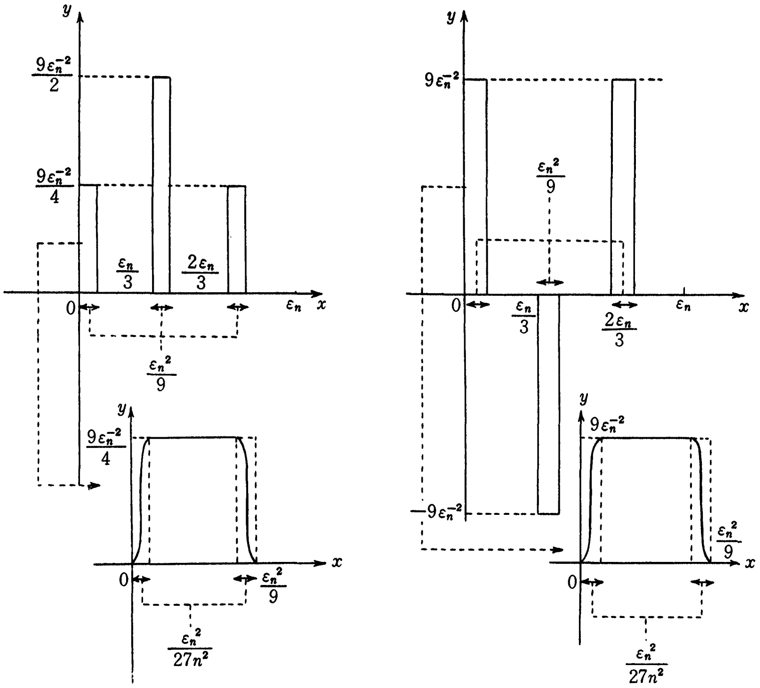

Fig. 1

$\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$, $\{\varphi_n^\delta\cdot\tilde{\varphi}_n^\delta\}$ *takes the following form;*

$$\{\varphi_n^\delta\cdot\tilde{\varphi}_n^\delta\}=\int_0^1 C(\alpha)\delta^{1+\alpha}d\mu(\alpha)+\sum_{n=0}^{\infty}C_n\delta^{(n)},$$

i.e. there exist $\varphi_n^\delta\in\mathfrak{c}(\delta\,|\,\tau_{D'}\,|\,\delta)$ and $\{\varphi_n\}\in\sum_{k=0}^{\infty}C_k\delta^{(k)}$ such that $\{\varphi_n^\delta\cdot\tilde{\varphi}_n^\delta\}=\int_0^1 C(\alpha)\{(\varphi_n^\delta)^{1+\alpha}\}\,d\mu(\alpha)+\{\varphi_n\}$.

Conversely, for any $C(\alpha)$ and C_k there exist $\{\varphi_n^\delta\}$, $\{\varphi_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ such that the above expression holds.

At the second step we give a range of product $[\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})\cdot\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})]_{(D')}$ for the case in which $\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ is the above set of sequences in this paragraph and the set of sequences $\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})$ has the following properties; there exists a sequence of positive number $\{\varepsilon_n\}$ such that $\lim_{n\to\infty}\varepsilon_n=0$ and $|\varphi_n^{1/x}-1/x|<\varepsilon_n$ for $x\geqq\varepsilon_n$ and $x\leqq-\varepsilon_n$. If $\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}$ defines a distribution, then its carrier is the original point, where

$$\{\varphi_n^\delta\},\ \{\psi_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta}),\ \{\varphi_n^{1/x}\},\ \{\psi_n^{1/x}\}\in\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x}).$$

So it must take the form $\sum_{k=0}^{N}C_k\delta^{(k)}$. If $\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}$ is not a distribution, it is expressed by the similar formula $\int_0^\infty C(\alpha)\delta^{1+\alpha}\,d\mu(\alpha)+\sum_{k=0}^{\infty}C_k\delta^{(k)}$ as in Theorem 1 [10]. Conversely, selecting suitable sequences, we can construct this expression for arbitrary C_k and $C(\alpha)$. Because, if the carrier of $\varphi_n^{1/x}$ is $x\geqq\varepsilon_n/2$ and $x\leqq-\varepsilon_n/2$, and if the carrier of φ_n^δ is $0\leqq x\leqq\varepsilon_n/2$, and if the carrier of $\varphi_n^{-\delta}$ is $-\varepsilon_n/2\leqq x\leqq 0$, then $\{\varphi_n^{1/x}+\varphi_n^\delta+\varphi_n^{-\delta}\}\in\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})$. Further if $\{\tilde{\varphi}_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ has the carrier in $0\leqq x\leqq\varepsilon_n/2$, we can construct $\{(\varphi_n^{1/x}+\varphi_n^\delta+\varphi_n^{-\delta})\cdot\tilde{\varphi}_n^\delta\}$ such that $\lim_{n\to\infty}(\varphi_n^{1/x}+\varphi_n^\delta+\varphi_n^{-\delta})\cdot\tilde{\varphi}_n^\delta=\lim_{n\to\infty}\varphi_n^{1/x}\cdot\tilde{\varphi}_n^\delta$ by the topology in $(\boldsymbol{D}')$. So we can apply the result of Theorem 1 for $\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ and $\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})$, and obtain the formula

$$\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}=\int_0^\infty C(\alpha)\delta^{1+\alpha}d\mu(\alpha)+\sum_{n=0}^\infty C_n\delta^{(n)}.$$

Theorem 2. *For* $\{\varphi_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ *and* $\{\varphi_n^{1/x}\}\in\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})$, $\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}$ *takes the following form;*

$$\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}=\int_0^\infty C(\alpha)\delta^{1+\alpha}d\mu(\alpha)+\sum_{k=0}^\infty C_k\delta^{(k)}.$$

Conversely for any $C(\alpha)$ and C_k there exist $\{\varphi_n^\delta\}$, $\{\varphi_n^{1/x}\}$ such that the above expression holds.

Next we select the topology τ_0 such that $\mathfrak{c}(\delta\,|\,\tau_0\,|\,\tilde{\delta})\subset\mathfrak{c}(\delta\,|\,\tau\,|\,\tilde{\delta})$ *and* $\mathfrak{c}(1/x\,|\,\tilde{\tau}_0\,|\,\widetilde{1/x})\subset\mathfrak{c}(1/x\,|\,\tilde{\tau}\,|\,\widetilde{1/x})$, by the following way;

(1) for all $\{\varphi_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau_0\,|\,\tilde{\delta})$ $x\varphi_n^\delta$ are uniformly bounded,

(2) for all $\{\varphi_n^{1/x}\}\in\mathfrak{c}(1/x\,|\,\tilde{\tau}_0\,|\,\widetilde{1/x})$ $x\varphi_n^{1/x}$ are uniformly bounded,

then we obtain the following

Theorem 3. (1) *For* $\{\varphi_n^\delta\},\{\tilde{\varphi}_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau_0\,|\,\tilde{\delta})$, $\{\varphi_n^\delta\cdot\tilde{\varphi}_n^\delta\}$ *takes the following form;*

$$\{\varphi_n^\delta\cdot\tilde{\varphi}_n^\delta\}=\int_0^1 C(\alpha)\delta^{1+\alpha}d\mu(\alpha)+\sum_{n=0}^1 C_n\delta^{(n)}.$$

(2) *For* $\{\varphi_n^\delta\}\in\mathfrak{c}(\delta\,|\,\tau_0\,|\,\tilde{\delta})$, $\{\varphi_n^{1/x}\}\in\mathfrak{c}(1/x\,|\,\tilde{\tau}_0\,|\,\widetilde{1/x})$, $\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}$ *takes the following form;*

$$\{\varphi_n^\delta\cdot\varphi_n^{1/x}\}=\int_0^\infty C(\alpha)\delta^{1+\alpha}d\mu(\alpha)+\sum_{n=0}^1 C_n\delta^{(n)}.$$

4. Güttinger's product. Güttinger defined the product of improper functions. We show here that his construction is the restricted one of our multiplication.

Definition. *We define the product* $\{\varphi_n^T\}\circ\{\varphi_n^S\}$ *of* $\{\varphi_n^S\}$ *and* $\{\varphi_n^T\}$ *by the following way* $\{\varphi_n^T\}\circ\{\varphi_n^S\}=\lim_{n\to\infty}(\lim_{m\to\infty}\{\varphi_n^T\cdot\varphi_m^S\})$, *and we define* $T\circ S=\lim_{n\to\infty}(\lim_{m\to\infty}\{\varphi_n^T\cdot\varphi_m^S\})$.

Lemma 2. *If the limit of the right hand side of definition of* $T\circ S$ *converges in* D' *for* $S=\delta^{(k)}$, *then* $\langle T\cdot\delta^{(k)},\varphi\rangle=\langle\sum_{i=0}^k C_i\delta^{(i)},\varphi\rangle$.

Remark. *This definition of the product is a part of the* **Definition 2.**

Proof. Let's decompose $\varphi(x) \in (\mathcal{D})$ to the following form; $\varphi(x) = \psi(x) + \sum_{i=0}^{k} \varphi^{(i)}(0)\phi_{(i)}(x)$, where $\phi_i(x)$ is a fixed function satisfying $D^m \phi_i(0) = \begin{cases} 1 & m=i \\ 0 & m \neq i. \end{cases}$ Then

$$\begin{aligned}\langle T\cdot\delta^{(k)}, \varphi(x)\rangle &= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \varphi_n^T \cdot \varphi_m^{\delta(k)}, \varphi(x)\rangle) \\ &= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \varphi_n^T \cdot \varphi_m^{\delta(k)}, \psi(x) + \sum_{i=0}^{k} \varphi^{(i)}(0)\phi_i(x)\rangle) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \{\langle \varphi_n^T \cdot \varphi_m^{\delta(k)}, \psi(x)\rangle + \sum_{i=0}^{k} \varphi^{(i)}(0)\langle \varphi_n^T \cdot \varphi_m^{\delta(k)}, \phi_i(x)\rangle\} \\ &= 0 + \sum_{i=0}^{k} \varphi^{(i)}(0) \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_n^T \cdot \varphi_m^{\delta(k)}, \phi_i(x)\rangle \\ &= \sum_{i=0}^{k} \varphi^{(i)}(0) \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_m^{\delta(k)}, \varphi_n^T \phi_i(x)\rangle.\end{aligned}$$

Conversely for any C_i $(i=1,\cdots,k)$ and T we can find $\{\varphi_n^{\delta(k)}\}$ $\{\varphi_n^T\}$ which satisfy the above relation as follows: $\varphi_m^{\delta(k)}$ has a carrier in an interval $(-\varepsilon_m, \varepsilon_m)$, where $\varepsilon_m > 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$, and $\varphi_n^T = \sum_{i=0}^{k} \left(\left[(-1)^k \binom{k}{i}\right]^{-1} C_i x^{k-i}/(k-i)!\right)$ in an interval $(-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

Lemma 3. *From the similar condition as that of* **Lemma 1**, *it follows that* $\langle T\cdot x^{-k}, \varphi\rangle = \langle S, \varphi\rangle + \langle \sum_{i=0}^{k-1} C_i \delta^{(i)}, \varphi(x)\rangle$, *where S is a distribution which has no more singularities than T at the origin.*

Proof. Let's decompose $\varphi(x) \in (\boldsymbol{D})$ to the following form; $\varphi(x) = \psi(x) + \sum_{i=0}^{m-1} \varphi^{(i)}(0)\phi_i(x)$, where $D^{(m)}\phi_i(0) = \delta_{mi}$. Then

$$\begin{aligned}\langle T\cdot x^{-m}, \varphi(x)\rangle &= \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_n^T \cdot \varphi_m^{x^{-k}}, \varphi(x)\rangle \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_n^T \cdot \varphi_m^{x^{-k}}, \psi(x) + \sum_{i=0}^{k-1} \varphi^{(i)}(0)\phi_i(x)\rangle \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \{\langle \varphi_n^T \cdot \varphi_m^{x^{-k}}, \psi(x)\rangle + \sum_{i=0}^{k-1} \varphi^{(i)}(0)\langle \varphi_n^T \cdot \varphi_m^{x^{-k}}, \phi_i(x)\rangle\} \\ &= S\varphi + \sum_{i=0}^{k-1} \varphi^{(i)}(0) \lim_{n\to\infty} \lim_{m\to\infty} \langle \varphi_m^{x^{-k}}, \varphi_n^T \phi_i(x)\rangle.\end{aligned}$$

Conversely for any C_i $(i=1,\cdots,k)$ and T we can find $\{\varphi_n^{x^{-k}}\}$, $\{\varphi_n^T\}$ which satisfy the above relation as follows: $\varphi_n^{x^{-k}} \cdot \varphi_n^T$ has the following properties;

$$(1) \qquad \int_{-\infty}^{-\varepsilon_n} \varphi_m^{x^{-k}} \varphi_n^T \phi_i(x)\, dx + \int_{\varepsilon_n}^{\infty} \varphi_m^{x^{-k}} \varphi_n^T \phi_i(x) dx = 0,$$

$$(2) \qquad \int_{-\varepsilon_n}^{\varepsilon_n} \varphi_m^{x^{-k}} \varphi_n^T \phi_i(x) dx = C_i,$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$, then $\langle T\cdot x^{-k}, \varphi\rangle = \langle S, \varphi\rangle + \langle \sum_{i=0}^{k-1} C_i \delta^{(i)}, \varphi(x)\rangle$. The condition (2) is satisfied if we select

$\{\varphi_m^{x-k}\}$ and $\{\varphi_n^T\}$ such that $\varphi_n^T \equiv C_t$ in the interval $(-\varepsilon_n, \varepsilon_n)$ and $\lim_{n\to\infty} \varphi_n^{x-k} = \sum_{i=0}^{k-1} \delta^{(i)}$ in the interval $(-\varepsilon_n, \varepsilon_n)$.

So we can obtain the conclusion of **Lemma 3.**

Güttinger defined the product AB of two distributions A and B. But, in his definition, B has essentially, only finite singular points and in other points $B \in C^\infty$. On these singular points the singularities of B are like $\delta^{(k)}$ or x^{-k}.

Now let's introduce the topology τ_B regarding to B such that $\mathfrak{c}(A \mid \tau \mid \widetilde{A})$, $\mathfrak{c}(B \mid \tau_B \mid \widetilde{B}) = [\{\varphi_n^B\}]$ satisfy the following condition:

Let $\{x_1, \cdots, x_N\}$ denote a set of singular points of B.

Let $U_{\varepsilon_n}\{x_1, \cdots, x_N\}$ denote a ε_n neighbourhood of $\{x_1, \cdots, x_N\}$, where $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

$$\sum_{p=0}^{\infty} | D^p(\varphi_n^B - B) | < \varepsilon_n \quad \text{for } x \notin U_{\varepsilon_n}\{x_1, \cdots, x_N\}.$$

Then we obtain following:

Theorem 4. *If we can construct a set of distributions* $\{C\} = \{C; C \in (\boldsymbol{D}'), C = \lim_{n\to\infty} (\lim_{m\to\infty} \{\varphi_n^A \cdot \varphi_n^B\})$ *in* $(\boldsymbol{D}')$, *where* $\{\varphi_n^A\} \in \mathfrak{c}(A \mid \tau \mid \widetilde{A})$ *and* $\{\varphi_n^B\} \in \mathfrak{c}(B \mid \tau_B \mid \widetilde{B})$, *then* $AB = C$ *in the sense of Güttinger's meaning.*

Proof. We can see this theorem from **Lemmas 2** and **3.**

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