71. Relations among Topologies on Riemann Surfaces. I

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Let R be a Riemann surface and let $R_n(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundary ∂R_n. Suppose R is a Riemann surface with positive boundary (if R has null-boundary, consider $R-R_0$ instead of R). Then we can introduce some topologies from the original topology (defined by local parameters) which are homeomorphic to the original topology in R. We know Stoilow's, Green's, K-Martin's and N-Martin's topologies[1] (we abbreviate them by $S.T$, $G.T$, $KM.\ T$ and $NM.\ T$ respectively in the present papers). Also we can define the ideal boundary B by the completion of R with respect to $\alpha\,(\alpha=S,\ G,\ KM$ or $NM)$-topology. When R is a subdomain in the z-plane, the boundary of R is realized. In this case also we can use the topology defined by Euclidean metric abbreviated by $E.T$. To study potential, analytic functions and the structure of Riemann surfaces, we use suitable topologies on R. But it is important to consider the relations among topologies on R.

Let $[p]^\alpha$ be a point of $\overline{R}=R+B$ with respect to α-topology and let $[v_n(p)]^\alpha=E\left[z\in\overline{R}: \overset{\alpha}{\text{dist}}\,(p,z)<\frac{1}{n}\right]$, where $\overset{\alpha}{\text{dist}}\,(p,z)$ is the distance between p and z with respect to α-topology. Suppose α and β-topologies are defined on $\overline{R}$. Then $\lim_n[\overline{v_n(p)}]^\alpha=p=\lim_n[\overline{v_n(p)}]^\beta$ for $p\in R$. If $\lim_n[\overline{v_n(p)}]^\alpha=[p]^\beta$ for every $p\in\overline{R}$, we say that α is finer than β and denote it by $\alpha\succ\beta$. If α is not finer than β and also β is not finer than α, we say that α and β are independent and denote it by $\alpha\succ\!\!\prec\beta$. Suppose $KM.\ T$ and $NM.\ T$ are defined in $\overline{R}$. Let B_1^γ be the set of γ-minimal point ($\gamma=K$ or N).[2] Then $B-B_1^\gamma=B_0^\gamma$ is an F_σ set of harmonic measure zero for K and of capacity zero for N respectively. Let G be a domain in R and $p\in B_1^\gamma$. If $K_{CG}(z,p)<K(z,p)$ $({}_{CG}N(z,p)<N(z,p))$, we say $G\overset{K}{\ni}p$ $(G\overset{N}{\ni}p)$, where $K_{CG}(z,p)$ $({}_{CG}(N(z,p))$ is the least positive super ($\overline{\text{super}}$[3]) harmonic function in R (in $R-R_0$) larger than G. Then we proved that such domains have almost the

1) Z. Kuramochi: On the behaviour of analytic functions on the ideal boundary. II, Proc. Japan Acad., **38**, 188-193 (1962).

2) Z. Kuramochi: On potentials on Riemann surfaces, Journ. Hokkaido Univ., (1962).

3) See 2).

same properties[4] of the system of neighbourhoods of p, i.e. 1) $v_n(p)\overset{\gamma}{\ni} p$. 2) If $G\overset{\gamma}{\ni} p$, int $(CG)\overset{\gamma}{\not\ni} p$. 3) If $G_i\overset{\gamma}{\ni} p$, $\bigcap\limits_{i=1}^{i_0} G\overset{\gamma}{\ni} p$. From above facts there exists at most one component of G' of G such that $G'\overset{\gamma}{\ni} p$. Suppose γ and β-topologies are defined on $\overline{R}$. If $\bigcap\limits_n \bar{G}_n=[p]^\beta$, we say that γ is finer γ-approximately than β at p and denote it by $\alpha\overset{\gamma}{\succ}\beta$, where $\underset{n}{G}\overset{\gamma}{\ni} p$ and the intersection is taken over all domains G_n such that $G_n\overset{\gamma}{\ni} p$ and the closure is taken with respect to γ-topology.

Relation S.T and other topologies. Since every neighbourhood $v_n(p)$ of $p\in B$ relative to $S.T$ has compact relative boundary, we see at once

Theorem 1. *$E.T\succ S.T$, $G.T\succ S.T$, $NM.T\succ S.T$ and $KM.T\succ S.T$.*

Approximate relations among topologies. We proved that $KM.\,T$, $NM.\,T$, $G.T$ and $S.T$ are $H.S$-topology (harmonically separative[5]) and $NM.\,T$, $G.T$ and $S.T$ are $D.S$ (Dirichlet-separative[5]) also it can be proved easily that $E.T$ is $H.S$ and $D.S$. Let $\underline{R}$ be a Riemann surface and let R be a covering surface over R and let $f(z)=w$: $z\in R$ and $w\in\underline{R}$ be an analytic function from R into $\underline{R}$. Then we proved[6]

Theorem 2. a). *Let R be a Riemann surface with $KM.T$ and $\underline{R}$ with an $H.S$-topology. Then if R is of positive boundary (if R is of null-boundary, $f(z)$ is a covering of F-type),*

$$\bigcap_n \overline{f(G_n)}=\textit{one point of } \underline{R}+\underline{B}$$

except a $G_{\delta\sigma}$ set of B of harmonic measure zero, where $G_n\overset{K}{\ni} p$.

Let $n(w)$ be the number of times when w is covered by R. If $n(w)\leqq M<\infty$ in a neighbourhood of $\underline{B}$ of $\underline{R}$ and there exists a neighbourhood (with respect to local parameter) $C(p)$ for $p\in\underline{R}$ such that every connected piece of R over $C(p)$ has finite area, we say that R is a covering surface almost finitely sheeted. Then

Theorem 2. b). *Let $\underline{R}$ be a Riemann surface (of positive boundary or of null-boundary) and let R be a covering surface (with $NM.\,T$) of almost finitely sheeted over $\underline{R}$ which has a $D.S$-topology.* Then

$$\bigcap \overline{f(G_n)}=\text{one point of } \underline{R}+\underline{B}\text{: } G_n\overset{N}{\ni} p$$

except a $G_{\delta\sigma}$ set of B of inner capacity zero.

Consider $R=\underline{R}$ and $w=f(z)$ an identical mapping and Martin's topologies and other topologies. Suppose $KM.T$ and another $H.S$ topology. Then by a) we have

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4) Z. Kuramochi: On the behaviour of analytic functions on the ideal boundary. I, Proc. Japan Acad., **38**, 150–155 (1962).

5) See 4).

6) Z. Kuramochi: On the behaviour of analytic functions on the ideal boundary. III, Proc. Japan Acad., **38**, 194–198 (1962).

Theorem 3. a). *KM. T$\overset{K}{\succ}$NM. T, KM. T$\overset{K}{\succ}$G.T, and KM. T$\overset{K}{\succ}$E.T except a $G_{\delta\sigma}$ set of harmonic measure zero.*

Similarly by b) we have

Theorem 3. b). *NM. T$\overset{N}{\succ}$G.T and NM.T$\overset{N}{\succ}$E.T except a $G_{\delta\sigma}$ set of inner capacity zero.*

Next we show *KM. T$\not\succ$NM. T, KM. T$\not\succ$E.T, NM. T$\not\succ$,E.T, KM. T$\not\succ$G.T* and *NM. T$\not\succ$G.T.*

M. Brelot[7] constructed the example of the following type

Example 1. Let C be a square: $0<Im\ z<6$ and $0<Re\ z<6$ and let

S_n^1: $Im\ z=\frac{6}{2^n}$, $0<Re\ z<1.5-a_n$,	S_n^1: $Im\ z=\frac{6}{2^n}$, $1<Re\ z<1.5-a_n$,
S_n^2: $Im\ z=\frac{6}{2^n}$, $1.5+a<Re\ z<4.5-a_n$,	S_n^2: $Im\ z=\frac{6}{2^n}$, $1.5+a_n<Re\ z<4.5-a_n$,
S_n^3: $Im\ z=\frac{6}{2^n}$, $4.5+a_n<Re\ z<5$,	S_n^3: $Im\ z=\frac{6}{2^n}$, $4.5+a_n<Re\ z<6$,
S_n: $Im\ z=\frac{6}{2^n}$, $0<Re\ z<5$,	S_n: $Im\ z=\frac{6}{2^n}$, $1<Re\ z<6$,
for odd number n.	for even number n.

Put $D=C-\sum_{n=1}^{\infty}S_n$. Then D is simply connected. Let $p_n^1=1.5+\frac{1}{2}\left(\frac{6}{2^n}+\frac{6}{2^{n+1}}\right)i$ and $p_n^2=4.5+\frac{1}{2}\left(\frac{6}{2^n}+\frac{6}{2^{n+1}}\right)i$ $(n=1,2\cdots)$. Map D onto $|\zeta|<1$. Then the images of $\{p_n^1\}$ and $\{p_n^2\}$ converge to the same point on $|\zeta|=1$. Whence $\lim_n K^D(z,p_n^1)=\lim_n K^D(z,p_n^2)$ for $z\in D$, i.e. $\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point relative to D. Put $\Omega=C-\sum_{n=1}^{\infty}(S_n^1+S_n^2+S_n^3)$. Then $\Omega\supset D$. He proved that $\{a_n\}$ can be chosen so small that Green's functions of D and Ω have almost same behaviour at $\{p_n^i\}(i=1,2)$ and proved that $\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point of Ω. Clearly $\{p_n^1\}$ and $\{p_n^2\}$ determine different accessible boundary points. Hence *KM. T$\not\succ$E.T*. This method can be also used for $N(z,p)$ and we have similarly *NM.T$\not\succ$E.T*. Also he constructed the example of the following type

Example 2. Let C be a circle, $|z|<1$. Let S: $\arg z=0, 0<z<1$. Let T_n: $\arg z=0$, $1-\frac{1}{2^n}<z<1-\frac{1}{2^n}+a_n\left(a_n<\frac{1}{4\times 2^n}\right)$. Let $\{p_n^1\}$ be sequences such that p_n^1: $|p_n^1-1|=\frac{1}{2^n}$, $\arg(p_n^1-1)=\frac{3\pi}{4}$ and p_n^2: $|p_n^2-1|=\frac{1}{2^n}$, $\arg(p-1)=-\frac{3\pi}{4}$.

7) M. Brelot: Sur le principe des singularités positives et la topologie de R. S. Martin, Ann. Univ. de Grenoble (1946).

Put $D=C-S$ and map D onto $|\zeta|<1$. Then $z=1$ is mapped two different points ζ_1 and ζ_2, whence $\lim_n K^D(z, p_n^1)\neq\lim_n K^D(z, p_n^2)$. Put $\Omega=C-S+\sum_1^\infty T_n$. We can choose a_n so small that $\{p_n^1\}$ and $\{p_n^2\}$ determine different K-Martin's points relative to Ω. On the other hand, clearly $\{p_n^1\}$ and $\{p_n^2\}$ determine the same boundary point. Hence $KM.T\not\prec E.T.$ Similarly as above it can be proved $NM.T\not\prec E.T.$ Thus we have

Theorem 4. a). *$KM.T\not\asymp E.T$ and $NM.T\not\asymp E.T$.*

Next we show $KM.T\not\asymp NM.T$.

Lemma 1. a). *Let F be a closed set in $C: |z|<1$ such that $F\cap\partial C=0$. Let $\Omega=C-F$ and $p_0: z=1$ on ∂C. Then there exists only one K-Martin's point of Ω on p_0.*

Let D be a simply connected domain in C such that $D\supset F$ and $\partial D\cap\partial C=0$. Put $v_n(p)=E\left[z: |z-p_0|<\frac{1}{n}\right]$. Let $G_n^\Omega(z, z_0)$ and $G_n(z, z_0)$ be Green's functions of $\Omega-v_n(p)$ and of $C-v_n(p)$: $z_0\in D-F$. We can suppose $v_n(p)\cap D=0$ for $n\geqq n_0$. Then $G_n^\Omega(z, z_0)=G_n(z, z_0)-H_n(z)$ in $C-D-v_n(p)$, where $H_n(z)$ is a positive harmonic function in $C-D-v_n(p)$ such that $H_n(z)=0$ on $\partial C+\partial v_n(p)$ and $H_n(z)=G_n(z, z_0)-G_n^\Omega(z, z_0)>0$ on ∂D. Since $\partial D\cap p_0=0$ and $\partial D\cap\partial C=0$, $H_n(z)\leqq M<\infty$ and $G_n^\Omega(z, z_0)>N>0$ on ∂D for $n\geqq n_0$. Hence there exists a constant L such that $H_n(z)<L\, G_n^\Omega(z, z_0)$ on ∂D, whence by the maximum principle $G_n^\Omega(z, z_0)\leqq G_n(z, z_0)\leqq G_n^\Omega(z, z_0)(1+L)$ in $C-D-v_n(p_0)$. Next by $G_n^\Omega(z, z_0)=G_n(z, z_0)=0$ on $\partial v_n(p)$ we have

$$0\leqq \frac{\partial}{\partial n} G_n(\zeta, z)\leqq(1+L)\frac{\partial}{\partial n} G_n^\Omega(\zeta, z) \text{ on } \partial v_n(p_0). \tag{1}$$

Let $V(z)$ be a positive harmonic function in Ω vanishing on $C-p_0+F$ (except a set of F of capacity zero). Then $V(z)=\lim_{n=\infty}\frac{1}{2\pi}\int V(\zeta)\frac{\partial}{\partial n}G_n^\Omega(\zeta, z)ds$. Let $U_n(z)$ be a positive least superharmonic function in C larger than $V(z)$ on $v_n(p_0)$. Then $U_n(z)\uparrow$. Put $U(z)=\lim_n U_n(z)$. Then $U(z)=\lim_n\frac{1}{2\pi}\int V(\zeta)\frac{\partial}{\partial n}G_n(\zeta, z)ds$. Hence by (1) $U(z)<\infty$. We say that $U(z)$ is obtained by extremisation[8] from $V(z)$ and denote $U(z)={}_{ex}V(z)$. Let $U(z)$ be a positive harmonic function vanishing on $C-p_0$. Let $V_n(z)$ be the least positive superharmonic function in $C-F$ larger than $U(z)$ in $v_n(p)$. Then $V_n(z)\downarrow V(z)$. We denote it by $V(z)={}_{inex}U(z)$. Then If ${}_{ex}V(z)<\infty$, ${}_{inex}({}_{ex}V(z))=V(z)$.[9] Assume there exist two K-Martin's points of Ω on p_0, i.e. $K^\Omega(z, p^1)\neq K^\Omega(z, p^2)$: p^i lies on p_0: $i=1, 2$. Then ${}_{ex}K(z, p^i)=0$ on $\partial C-p_0$. Now C is a unit

8) Z. Kuramochi: Relations between harmonic dimensions, Proc. Japan Acad., **34**, 576-580 (1954).

circle and there exists ϕ only one positive harmonic function $K^C(z, p_0)$ which vanishes on $\partial C - p_0$. Whence ${}_{ex}K^\Omega(z, p^i)$ is a multiple of $K^C(z, p_0)$, i.e. ${}_{ex}K^\Omega(z, p^i) = a_i K^C(z, p_0)$ and $K^\Omega(z, p^i) = a_i \ {}_{inex}(K^C(z, p_0))$. But $K^\Omega(q, p^1) = K^\Omega(q, p^2) = 1$: q is a fixed point in Ω and $a_1 = a_2$. This contradicts $K^\Omega(z, p^1) \neq K^\Omega(z, p^2)$. Hence we have Lemma 1. Similarly as above we have

Lemma 1. b). *Let D be a Riemann surface with positive boundary and let $\{p_n^1\}$ and $\{p_n^2\}$ be two sequences determining the same K-Martin's point of D. Let F be a compact set in D and put $\Omega = D - F$. Then $\{p_n^1\}$ and $\{p_n^2\}$ also determine the same K-Martin's point relative to Ω.*

Let S be a sector such that $1 < |z| < \exp \gamma$, $0 < \arg z < \Theta$ with a finite number of radial slits. Let $U(z)$ be a harmonic function in S with boundary value $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ on $|z| = 1$ and $|z| = \exp \gamma = r$, where $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ are continuous. Then $D(U(z)) \geqq \frac{1}{2\pi\gamma}\int_0^\Theta |\varphi(e^{i\theta}) - \varphi(re^{i\theta})|^2 d\theta$.[9] Map S by $\zeta = \frac{\log z}{\Theta}$ onto $0 < Re\,\zeta < 1, 0 < Im\,\zeta < \gamma$. Then we have

Lemma 2. *Let $U(z)$ be a harmonic function in a rectangle $0 < Re\,z < 1, 0 < Im\,z < \gamma$ with a finite number of vertical slits and suppose $U(z)$ is continuous on $Im\,z = 0$ and $Im\,z = \gamma$. Then*

$$D(U(z)) \geqq \frac{1}{\gamma}\int_0^1 |U(x+iy) - U(x)|^2 ds.$$

Lemma 3. *Let C be a circle $|z| < 1$ and let Λ be an arc on ∂C: $|\arg z| < a$. Let Γ be analytic curve such that $\Gamma \cap v_\delta(p_0) = 0$ for a number $\delta > 0$: $v_\delta(p_0) = E[z: |z-1| < \delta]$. Then for any given positive number ε there exists a number a (depending on Γ and z_0) such that $\frac{w(z, \Lambda, C)}{G(z, z_0)}$ on Γ for length of $\Lambda < a$, where z_0 is a point such that $z_0 \notin v_\delta(p)$, $G(z, z_0)$ and $w(z, \Lambda, C)$ is a Green's function and harmonic measure of Λ relative to C.*

Map $|z| < 1$ by a linear transformation $\zeta = \zeta(z)$ onto $|\zeta| < 1$ so that $z = 1 \to \zeta = 1$, $z = z_0 \to \zeta = 0$. Then $\Lambda \to \Lambda^*$ and $\Gamma \to \Gamma^*$ respectively. Now $w(\zeta, \Lambda^*, C) = \frac{1}{2\pi}\int_{\Lambda^*} \frac{(1-r^2)}{1 - 2r\cos(\theta - \varphi) + r^2} d\varphi$: $\zeta = re^{i\theta}$. We can suppose $v_{\delta_1}(p_0) \cap \Gamma^* = 0$ for $\delta_1 > 0$ and $|\arg \zeta| > \delta_2 > 0$ for $\zeta \in v_{\delta_1}(1)$. Hence $w(\zeta, \Lambda^*, C) \leqq \frac{\text{length of}}{2\pi \sin^2 \delta_2}(1 - r^2)$ on Γ^*. On the other hand, $G(z, z_0) = -\log r$: $r = |\zeta|$. Hence we can choose a number a such that $w(z, \Lambda, C) \leqq \varepsilon G(z, z_0)$ on Γ.

Lemma 4. *Let D be a Riemann surface with positive boundary*

9) See 2).

and let $E^1 \supset E^2$ *and* $F^1 \supset F^2$ *be closed sets in* D. *Let* $G_M^L(z, z)$ *be a Green's function of* $(D-M)-L$ *with pole* z_0 *in* $D-L-M(G^L(z, z_0)$ *means simply that of* $D-L$). *Then*

$$G_{E^2}^{F^2}(z, z_0)-G_{E_2}^{F^1}(z, z_0)\geqq G_{E^1}^{F^2}(z, z)-G_{E^1}^{F^1}(z, z). \tag{2}$$

Let $F_i^1 \supset F_i^2 (i=1, 2, \cdots, n)$ *be closed sets in* D. *Then*

$$G^{\sum_i F_i^2}(z, z_0)-G^{\sum_i F_i^1}(z, z_0)\leqq \sum_i (G^{F_i^2}(z, z_0)-G^{F_i^1}(z, z_0)). \tag{3}$$