# 66. A Note on the First Boundary Value Problem on Martin Spaces Induced by Markoff Chains

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1. The first boundary value problem on Martin spaces induced by Markoff chains has been studied by J. L. Doob [1], T. Watanabe [6] and G. A. Hunt [4]. Their arguments are based on the probabilistic interpretation of the theory of R. S. Martin and the martingale convergence theorem established by J. L. Doob. In this paper, we state simple remarks for the conditions that any boundary point of the Martin space induced by a Markoff chain is regular with respect to the first boundary value problem. Our argument depends on the properties of sojourn solutions studied by W. Feller [2], but not on the martingale convergence theorem. The author wishes to express his thanks to Prof. S. Tsurumi, Mr. K. Sato and Prof. T. Watanabe for their valuable advice.

2. Let $x(t)$ be a temporally homogeneous Markoff chain on a countable set $S$ with continuous time. We suppose that $x(t)$ is minimal in the sense of [3] and all points of $S$ are transient and we consider a fixed reference measure $\gamma$, introduced by G. A. Hunt [4]. We concern with the Martin space $M(=S+\partial S)$, with its topology $\rho$, induced by $x(t)$ with respect to $\gamma$ as in H. Kunita and T. Watanabe [5]. We shall denote by $B(\partial s)$ the Borel field consisting of all $P$-Borel subsets in $\partial S$.

Let $\mathrm{P}(x, \cdot)$ be the probability with the initial mass at $x \in S$ and let $z(i-0)=\lim_{n\to+\infty} z(n)$, where $z(n)$ is the $n$-th jumping time, then the harmonic measure $h(x, B)$ from $x$ to $B \in B(\partial S)$ is, in our case,

$$h(x, B)=P(x, x(z(i-0)) \in B).$$

For any set $A \subset S$,

$$S_A(x)=P(x, \bigcup_{k\geq 1} \bigcap_{n\geq k} (x(z(n)) \in A))$$

is called a sojourn solution and $A$ a sojourn set, if $S_A(\gamma)>0$. The following Lemma 1 shows the close relation between harmonic measures and sojourn solutions.

We introduce here some notations:

$(\partial S)_F=(b; b \in \partial S, \mathrm{h}(\gamma, G)>0$ for any open neighborhood $G$ of $b.)$.

For any $b \in \partial S$ and $\alpha, \beta>0$, let

$$\partial G_\alpha^b=(b'; b' \in \partial S, \rho(b', b) \leq \alpha),$$
$$G_\alpha^b=(x; x \in S, \rho(b, x) \leq \alpha),$$

$$\partial C_\alpha^b = \partial S - \partial G_\alpha^b,$$
$$\partial U_\beta^\alpha = (b';\, b' \in \partial S,\ \rho(b', \partial C_\alpha^b) \leq \beta),$$
$$U_\beta^\alpha = (x;\, x \in S,\ \rho(x, \partial C_\alpha^b) \leq \beta).$$

For simplicity, we shall treat only the case where $(\partial S) = (\partial S)_F$ in the following arguments.

Lemma 1. For any set $B \in B(\partial S)$ for which $h(\gamma, B) > 0$ there exists a sojourn set $A \subset S$ such thal $h(x, B) = S_A(x)$ $x \in S$.

Proof. First suppose that $B$ is closed. For any $\alpha > 0$, consider $\partial D_\alpha$ and $D_\alpha$ defined by

$$\partial D_\alpha = (b;\, b \in \partial S,\ \rho(b, B) \leq \alpha),$$
$$D_\alpha = (x;\, x \in S,\ \rho(x, B) \leq \alpha).$$

Clearly $D_\alpha$ is a sojourn set and $D_\alpha$ tends to $B$ as $\alpha$ tends to 0. Moreover, there exists at least one countable set $(\alpha_i) \subset (0, 1)$ such that

$$\alpha_i \downarrow 0 \qquad (i \to +\infty),$$
$$h(x, \partial D_{\alpha_i}) = S_{D_{\alpha_i}}(x) \qquad x \in S.$$

In fact, for every $x \in S$, we define

$$N_B^x = (\alpha;\, \alpha \in (0, 1),\ P(x, x(z(i-0)) \in (\partial D_\alpha - (\partial D_\alpha)^i)) > 0),$$

then it is clear that each $N_B^x$ is a countable set, and that, for any $\alpha \notin N = \bigcup_{x \in S} N_B^x$,

$$P(x, x(z(i-0)) \in \partial D_\alpha) = P(x, \bigcup_{k \geq 1} \bigcap_{n \geq k} (x(z(n)) \in D_\alpha)) \ x \in S.$$

Since $h(x, \partial D_{\alpha_i})$ decreases to $h(x, B)$, by Theorem 9.3 in Feller's paper [2] (from now on, we shall denote this by ([2] Theorem 9.3), $h(x, B)$ is a sojourn solution.

For any open set and then for any general Borel subset of $S$, the above assertion may be verified by using the usual approximation methods and ([2] Theorem 9.3).

Finally, let $A_\varepsilon$ for any $\varepsilon (0 < \varepsilon < 1)$ be

$$A_\varepsilon = (x;\, x \in S,\ h(x, B) > 1 - \varepsilon),$$

then by ([2] Lemma 9.1),

$$h(x, B) = S_{A_\varepsilon}(x) \qquad x \in S,$$

the assertion is proved.

By using the similar results as that proved in Lemma 1, we shall obtain the following:

Lemma 2. Let $M^b \subset (0, 1)$ be, for any $b \in \partial S$,

$$M^b = (\alpha;\, \alpha \in (0, 1),\ h(x, \partial G_\alpha^b) \neq S_{G_\alpha^b}(x) \text{ for some } x \in S),$$

then $M^b$ is at most countable. And let $N_\alpha^b \subset (0, 1)$ be, for any $b \in \partial S$ and $\alpha > 0$,

$$N_\alpha^b = (\beta;\, \beta \in (0, 1),\ h(x, \partial U_\beta^\alpha) \neq S_{U_\beta^\alpha}(x) \text{ for some } x \in S),$$

then $N_\alpha^b$ is at most countable.

Theorem 1. For any $b \in \partial S$ and $\partial G_\alpha^b$, the following conditions are mutually equivalent.

(A) $$\lim_{x\to b} h(x, \partial G_\alpha^b)=1,$$

(B) $$\lim_{x\to b} h(x, \partial G_\alpha^b)=1 \text{ exists},$$

(C) let the set $A_\varepsilon$ for any number $\varepsilon(0<\varepsilon<1)$ be

$$A_\varepsilon=(x; x\in S, h(x, \partial G_\alpha^b)>1-\varepsilon),$$

then there exists $\alpha_0>0$ such that

$$G_{\alpha_0}^b\subset A_\varepsilon.$$

Proof. Since (C)$\Rightarrow$(A)$\Rightarrow$(B) is clear, it suffices to show (B)$\Rightarrow$(C). Suppose that there exist $\varepsilon_0$ $(0<\varepsilon_0<1)$ and a fundamental sequence $(y_m^b)$, which tends to $b$ as $m$ tends to infinity, such that

$$(y_m^b)\subset S-A_{\varepsilon_0}.$$

Then it follows that $(y_m^b)\subset U_{\varepsilon_1}$, where

$$U_{\varepsilon_1}=(x; x\in S, h(x, \partial C_\alpha^b)>1-\varepsilon_1),$$

$$\varepsilon_1=1-\varepsilon_0.$$

By the condition (B), for any $\varepsilon(\varepsilon_1<\varepsilon<1)$, there exists a number $\beta_0$ such that

$$h(x, \partial C_\alpha^b)>1-\varepsilon$$

holds for any $x\in G_\beta^b$ with $\beta<\beta_0$. Let $\beta$ and $\beta'$ be so small that

$$\rho(G_\beta^b, U_{\beta'}^\alpha)>0,$$

and $\beta\notin M^b$, $\beta'\notin N_\alpha^b$. Then

$$S_{U_{\beta'}^\alpha}(x)=h(x, \partial U_{\beta'}^\alpha)\geq h(x, \partial C_\alpha^b)=S_{U_\varepsilon}(x),$$

and so, by ([2] Lemma 9.2),

$$S_{\widetilde{U}_\varepsilon}(x)=S_{U_\varepsilon}(x),$$

where $\widetilde{U}_\varepsilon=U_\varepsilon\cap U_{\beta'}^\alpha$. Let $(G_\beta^b)$ be the representative sojourn set of $G_\beta^b$. Then, by ([2] Lemma 8.2), we have

(1) $$S_{U_\varepsilon}(x)=S_{(G_\beta^b)}(x)+S_{(U_\varepsilon-(G_\beta^b))}(x).$$

Using ([2] Lemma 4.2) and ([2] Lemma 8.2) again, we have, by simple calculations,

(2) $$S_{U_\varepsilon}(x)=S_{\widetilde{U}_\varepsilon}(x)\cap S_{U_\varepsilon}(x)=S_{(U_\varepsilon-(G_\beta^b))}(x).$$

Combining (1) and (2), it follows that

$$S_{(G_\beta^b)}(x)=S_{G_\beta^b}(x)=h(x, \partial G_\beta^b)=0 \qquad x\in S.$$

This contradicts to our assumption $(\partial S)=(\partial S)_F$. Thus the assertion is proved.

Theorem 2. Let $b$ be any fixed boundary point. The conditions (A), (B), and (C) for any $\partial G_\alpha^b$ in the preceding theorem are equivalent to the following condition: for any continuous function $f$ on $\partial S$, there exists an $x(t)$-harmonic function $u$ on $S$ such that

$$\lim_{x\to b} u(x)=f(b).$$

Proof. Suppose that (A) for any $\partial G_\alpha^b$ holds. Let a function $u$ on $S$ be

$$u(x)=\int_{\partial S} f(b)\, h(x, db),$$

then it is easily seen that $u$ is $x(t)$-harmonic on $S$ by virtue of the main representation theorem of $x(t)$-harmonic functions in the theory of Martin boundary (cf. [1], [4], [7]).

Since $f$ is continuous on $\partial S$, for any $\varepsilon>0$, there exists $\alpha>0$ such that

$$|f(b')-f(b)|<\varepsilon$$

for any $b'\in\partial G_\alpha^b$. Thus

$$\begin{aligned}|u(x)-f(b)|&\leq\int_{\partial G_\alpha^b}|f(b')-f(b)|\,h(x,db')\\&\quad+\int_{\partial C_\alpha^b}|f(b')-f(b)|\,h(x,db'),\\&\leq\varepsilon h(x,\partial G_\alpha^b)+2\,||f\,||_\infty h(x,\partial C_\alpha^b).\end{aligned}$$

Letting $x\to b$, we have

$$\overline{\lim_{x\to b}}\,|u(x)-f(b)\leq\varepsilon.$$

Since $\varepsilon$ is arbitrary, the assertion follows.

The proof of the inverse part of the theorem is clear.

3. We introduce here the generalized Poisson kernels $K(\gamma,x,b)$ for any $x\in S$ and $b\in\partial S$:

$$K(\gamma,x,b)=\lim_{y\to b}G(x,y)/G(\gamma,y),$$

where $G(x,y)=\sum_{n\geq0}P(x,x(z(n))=y)$. Note that the above limit exists by the very definition of the Martin boundary. We shall present the sufficient condition for (A) in Theorem 1 by using the properties of the generalized Poisson kernels.

We may readily verify the following:

Theorem 3. 1) Let $b$ be any fixed boundary point. The following condition $(D_1)+(D_2)$ is a sufficient condition for (A).

$(D_1)$ For any fixed $\partial G_\alpha^b$, there exist $\alpha_0>0$ and $K>0$, which depend only on $\alpha$, such that, for any $x\in G_{\alpha_0}^b$,

$$K(\gamma,x,b')\leq K<+\infty$$

holds for $h(\gamma,\cdot)$-almost all $b'\in\partial C_\alpha^b$,

$(D_2)$ $\lim_{x\to b}K(\gamma,x,b')$ exists for $h(\gamma,\cdot)$-almost all $b'\in\partial C_\alpha^b$.

2) If $(D_1)$ is satisfied, then $(D_2)$ and the following condition $(D_3)$ are mutually equivalent.

$(D_3)$ $\lim_{x\to b}K(\gamma,x,b')=0$ for $h(\gamma,\cdot)$-almost all $b'\in\partial C_\alpha^b$.

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