106. A Note on the Cut Extension of C-Spaces

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§1. Let R be *a semi-ordered linear space* which is *Archimedean*.[1] A semi-ordered linear space $\hat{R}$ is called the *cut extension of* R, if there exists a mapping of R into $\hat{R}$ ($R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$) such that

(C. 1) $(\alpha a+\beta b)^{\hat{R}}=\alpha a^{\hat{R}}+\beta b^{\hat{R}}$ *for any* $a, b \in R$ *and real numbers* α, β;

(C. 2) $a \leqq b$ *if and only if* $a^{\hat{R}} \leqq b^{\hat{R}}$;

(C. 3) $\bigcap_{\lambda \in \Lambda} a_\lambda=0$ $(a_\lambda \in R, \lambda \in \Lambda)$ *implies* $\bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}=0$ *in* $\hat{R}$;

(C. 4) $\hat{R}$ *is universally continuous*;[2]

(C. 5) *for each* $\hat{a} \in \hat{R}$ *there exists a system of elements* $a_\lambda \in R$ $(\lambda \in \Lambda)$ *such that* $\hat{a}=\bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$.

When we consider R as a lattice, $\hat{R}$: the *cut extension* of lattice R is nothing but a *normal completion* of R in Birkhoff's terminology [1].

It is well known ([4], Theorems 30.2 and 30.3) that *for any Archimedean semi-ordered linear space R there exists always $\hat{R}$: the cut extension of R, and $\hat{R}$ is determined uniquely up to an isomorphism.*

Now let E be a *compact Hausdorff space* throughout this paper and $C(E)$ be the space of all continuous functions defined on E. $C(E)$ is a semi-ordered linear space (by the usual addition and order) which is not always continuous, but Archimedean [2, 5, 6]. Thus, as is shown above, $\widehat{C(E)}$: the *cut extension* of $C(E)$ may be considered. The structure of $\widehat{C(E)}$ was investigated in [2] and it was proved that *$\widehat{C(E)}$ is isomorphic to the C-space $C(\mathcal{E})$, where $\mathcal{E}$ is the Boolean space associated with the lattice of regularly open sets*[3] *in* E, while $\mathcal{E}$ comes to be different from the original space E in most cases.

The aim of this note is to construct a function space on E which is isomorphic to $\widehat{C(E)}$. The result is the following:

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1) R is called *Archimedean*, if $\bigcap_{\nu=1}^{\infty} \frac{1}{\nu} a=0$ for every $0 \leqq a \in R$.

2) A semi-ordered linear space is called *universally continuous*, if for any bounded system of elements : $\{a_\lambda : a_\lambda \leqq a, \lambda \in \Lambda\}$ there exists $\bigcup_{\lambda \in \Lambda} a_\lambda$.

3) A subset G of E is called to be *regularly open*, if $G^{-o}=G$.

$\widehat{C(E)}$ is isomorphic (as a semi-ordered linear space) to $C_q(E)/N$, where $C_q(E)$ is a space of all bounded quasi-continuous functions on E and N is a linear manifold of $C_q(E)$ consisting of all f such that $f(x)=0$ for all $x \in A_f'$ (A_f is a set of the first category in E which depends on f).[4]

R. P. Dilworth proved in [2] that the normal completion of $C(E)$ (considered as a lattice) is lattice-isomorphic to the set of normal upper semi-continuous functions[5] on E. This can not be, however, regarded as a cut extension of $C(E)$, because the set of normal upper semi-continuous functions does not constitute a linear space in general.

§2. Let $B(E)$ be the totality of all bounded real functions on E. For any $f \in B(E)$ we denote by f^* (f_*) the *upper function* (resp. *lower function*) of f:

$$f^*(x)=\inf_U \{\sup_{y \in U} f(y)\}. \qquad (x \in E)$$
$$f_*(x)=\sup_U \{\inf_{y \in U} f(y)\},$$

where U runs over all neighbourhoods of x. An element $f \in B(E)$ is called *quasi-continuous* [3], if $((f^*)_*)^*=(f_*)^*$, and the totality of all bounded quasi-continuous functions on E is denoted by $C_q(E)$. The following lemma is due to H. Nakano ([3], Theorem 4 and 11 in §63).

Lemma 1. *$f \in B(E)$ is quasi-continuous if and only if f is continuous at each point of the complement of a set of the first category, and $C_q(E)$ is an Archimedean semi-ordered linear space including $C(E)$ by the usual addition and order.*

Now let N be the set of all $f \in C_q(E)$ such that $f(x)=0$ holds for all $x \in E$ except a set of the first category which depends on f. Since N is clearly a *semi-normal manifold*[6] of $C_q(E)$, $C_q(E)/N$ comes to be also a semi-ordered linear space and we denote by $\dot{f}$ an element of $C_q(E)/N$, i.e. a *residue class by N*, and denote also $C_q(E)/N$ by $\boldsymbol{C}_q(E)$.

For any $f \in C(E) \subset C_q(E)$, $f^{\hat{c}}$ denotes an element of $\boldsymbol{C}_q(E)$ to which f belongs.[7] Since the set $\{x : f(x) \neq g(x)\}$ ($f \neq g, f, g \in C(E)$) is open and not of the first category, $f \neq g$ ($f, g \in C(E)$) implies $f^{\hat{c}} \dot{\neq} g^{\hat{c}}$.

We shall show in the sequel that $\boldsymbol{C}_q(E)$ is isomorphic to the cut extension of $C(E)$. Since it is clear that the mapping: $C(E) \ni g \to g^{\hat{c}} \in \boldsymbol{C}_q(E)$ satisfies (C.1) and (C.2) in §1, we shall prove that it does also

4) For any $A \subset E$, A' denotes the complement of A. Since E is compact, EA_f' is dense in E.

5) A bounded function f is called to be *normal upper semi-continuous*, if $(f_*)^*=f$.

6) A linear manifold M of a semi-ordered linear space is called *semi-normal*, if $|b| \leqq |a|$, $a \in M$ implies $b \in M$.

7) Let ι be the inclusion mapping: $C(E) \xrightarrow{\iota} C_q(E)$ and q be the quotient mapping: $C_q(E) \xrightarrow{q} C_q(E)/N$. Then $f^{\hat{c}}=q(\iota(f))$ for $f \in C(E)$.

the conditions (C.3), (C.4) and (C.5) in the following lemmas.

Lemma 2. *For any* $\dot{0} \lneqq \dot{f} \in \boldsymbol{C}_q(E)$[8] *there exists* $0 \lneqq f \in C(E)$ *such that* $\hat{f^c} \dot{\leqq} \dot{f}$ *and the mapping*: $C(E) \ni g \to \hat{g^c} \in \boldsymbol{C}_q(E)$ *satisfies* (C.3).

Proof. Let $f \in C_q(E)$ be an arbitrary element belonging to $\dot{f}$. By virtue of Lemma 1 there exists a set A of the first category such that f is continuous and $f(x) \geqq 0$ on EA'. As $\dot{0} \dot{\lneqq} \dot{f}$, there exist $\varepsilon > 0$ and a open set $O \neq \phi$ such that $\{x : f(x) > \varepsilon\} \supseteqq OA'$. Let $x_0 \in O$ and $a(x)$ be a continuous function on E satisfying

i) $a(x_0) = \varepsilon$, $0 \leqq a(x) \leqq \varepsilon$ for all $x \in E$;

ii) $a(x) = 0$ for all $x \in O'$.[9]

For this $a \in C(E)$ we have $a(x) \leqq \varepsilon < f(x)$ for all $x \in OA'$ and $a(x) = 0 \leqq f(x)$ on $O'A'$. From this it follows that $a(x) \leqq f(x)$ holds for all $x \in EA'$, hence $\hat{a^c} \dot{\leqq} \dot{f}$ holds. The remainder of this lemma is the direct consequence of this fact. Q.E.D.

Lemma 3. *For any bounded system*[10] $\{\varphi_\lambda : \varphi_\lambda \in C(E), \lambda \in \Lambda\}$ *of continuous functions, putting* $f_0(x) = \sup_{\lambda \in \Lambda} \varphi_\lambda(x)$ $(\lambda \in \Lambda)$, *we obtain a quasi-continuous function* $f_0 \in C_q(E)$ *for which* $\dot{f_0} \doteq \bigcup_{\lambda \in \Lambda} \hat{\varphi_\lambda^c}$ *holds in* $\boldsymbol{C}_q(E)$.

Proof. It follows from the definition of f_0 that f_0 is lower semi-continuous, hence quasi-continuous by virtue of Theorem 2 of §63 in [3]. As $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is a bounded system, f_0 is also evidently bounded and $f_0 \in C_q(E)$. $f_0(x) \geqq \varphi_\lambda(x)$ for every $x \in E$ and $\lambda \in \Lambda$ implies $f_0 \geqq \varphi_\lambda$ $(\lambda \in \Lambda)$ and also $\dot{f_0} \dot{\geqq} \hat{\varphi_\lambda^c}$ $(\lambda \in \Lambda)$. Conversely let $\dot{g}$ be an element of $\boldsymbol{C}_q(E)$ for which $\dot{g} \dot{\geqq} \hat{\varphi_\lambda^c}$ $(\lambda \in \Lambda)$ holds. If $\dot{f_0} - \dot{f_0} \frown \dot{g} \dot{\gneqq} 0$, there exists $0 \lneqq h \in C(E)$ such that $\dot{0} \lneqq \hat{h^c} \dot{\leqq} \dot{f_0} - \dot{f_0} \frown \dot{g}$ by virtue of the above lemma. It follows from above $\hat{h^c} + \dot{f_0} \frown \dot{g} \dot{\leqq} \dot{f_0}$ and $\hat{\varphi_\lambda^c} \dot{\leqq} \dot{f_0} \frown \dot{g}$ $(\lambda \in \Lambda)$, hence $\hat{h^c} + \hat{\varphi_\lambda^c} \dot{\leqq} \dot{f_0}$. Since f_0 is quasi-continuous, there exists a set A of the first category such that f_0 is continuous at each point of EA'. Now we have from above

$$h(x) + \varphi_\lambda(x) \leqq f_0(x) \text{ for all } x \in EA' \text{ and } \lambda \in \Lambda,$$

because for any $\lambda \in \Lambda$ $h(x) + \varphi_\lambda(x) \leqq h_0(x)$ holds for every $x \in EA'B'_\lambda$, where B_λ is a set of the first category and $EA'B'_\lambda$ is dense in EA'.[11] As $\lambda \in \Lambda$ is arbitrary, it follows from above that $h(x) + \sup_{\lambda \in \Lambda} \varphi_\lambda(x) \leqq f_0(x)$, whence $h(x) = 0$ on EA', which contradicts the assumption that $h \gneqq 0$. Therefore we have $\dot{f_0} - \dot{f_0} \frown \dot{g} \doteq 0$, i.e. $\dot{f_0} \dot{\leqq} \dot{g}$, consequently $\dot{f_0} \doteq \bigcup_{\lambda \in \Lambda} \hat{\varphi_\lambda^c}$.

Q.E.D.

Lemma 4. *For every* $\dot{f} \in \boldsymbol{C}_q(E)$ *there exists a system of continu-*

8) We denote by $\doteq$, $\dot{\leqq}$ the equal relation and the order relationi n $\boldsymbol{C}_q(E)$ respectively.

9) Since E is compact, E is *completely regular*.

10) This means that $\varphi_\lambda \leqq f$ $(\lambda \in \Lambda)$ for some $f \in C(E)$.

11) Since $A \smile B_\lambda$ is of the first category and E is compact, $A'B'_\lambda$ is also dense in E.

ous functions $\{\varphi_\lambda \in C(E), \lambda \in \Lambda\}$ *such that* $\dot{f} = \bigcup_{\lambda \in \Lambda} \hat{\varphi}^{c}_{\lambda}$, i.e. *the mapping* $C(E) \ni f \to \hat{f}^{c} \in C_q(E)$ *satisfies* (C.5).

Proof. Let F be a set: $\{f : f \in C(E), \hat{f}^{c} \dot{\leqq} \dot{f}\}$. As f is bounded on E for an arbitrary $f \in \dot{f}$, F is a bounded system of continuous functions and f_0 $(f_0(x) = \sup_{f \in F} f(x)$ $(x \in E))$ is a quasi-continuous function with $\dot{f}_0 \leqq \dot{f}$ by Lemma 3. If $\dot{f} \gneqq \dot{f}_0$, then there exists $0 \lneqq f \in C(E)$ for which $\dot{f} - \dot{f}_0 \geqq \hat{f}^{c}$ holds. In view of the construction of F and f_0 we obtain a contradiction by the same way as the proof of the preceding lemma, whence we have $\dot{f} \doteq \dot{f}_0 \doteq \bigcup_{f \in F} \hat{f}^{c}$. Q.E.D.

Lemma 5. $C_q(E)$ *is untiversally continuous.*

Proof. Let $\dot{f}_\lambda \leqq \dot{f}$ $(\lambda \in \Lambda)$ and F be the set $\{f : f \in C(E), \hat{f}^{c} \dot{\leqq} \dot{f}_\lambda$ for some $\lambda \in \Lambda\}$, then F is a bounded system in $C(E)$ and by virtue of Lemma 3 there exists $f_0 \in C_q(E)$ for which $\dot{f}_0 \doteq \bigcup_{f \in F} \hat{f}^{c}$ holds. From this and the above lemma we may infer easily that $\dot{f}_0 \doteq \bigcup_{\lambda \in \Lambda} \dot{f}_\lambda$. Therefore $\boldsymbol{C}_q(E)$ is universally continuous. Q.E.D.

Collecting the results of Lemma 1~Lemma 5, we obtain

Theorem 1. *The cut extension of* $C(E)$ *is isomorphic to* $\boldsymbol{C}_q(E) = C_q(E)/N$.

§3. An element $f \in B(E)$ is called *normal quasi-continuous*, if $f_* = (f^*)_*$ and $f^* = (f_*)^*$ hold. Let $\mathfrak{M}$ be the totality of all normal quasi-continuous functions belonging to $C_q(E)$. It is evident that $C(E) \subset \mathfrak{M} \subset C_q(E)$, hence we may consider $\mathfrak{M}/N$, i.e. *the image* $q(\mathfrak{M})$ *of* $\mathfrak{M}$ by the quotient mapping $q : C_q(E) \xrightarrow{q} C_q(E)/N$. For each $f \in C_q(E)$, let $f^{\times}$ be defined by the formula:

$$f^{\times}(x) = \inf_{x \in U} \{\sup_{y \in UA'} f(x)\} \quad (x \in E),$$

where A is a set of the first category such that f is continuous on A'. Then $f^{\times}$ is *normal upper semi-continuous* ([3], Theorem 17 in §62), i.e. $((f^{\times})_*)^* = f^{\times}$ and obviously $f^{\times} \in \mathfrak{M}$ with $f^{\times} - f \in N$. Therefore we may see that $\mathfrak{M}/N = C_q(E)/N$ holds. From this and Theorem 1 it follows that $\mathfrak{M}/N$ *is a universally continuous semi-ordered linear space.*

Now we introduce an equivalent relation (I) in $\mathfrak{M}$ as follows:

$$f \overset{def.}{\equiv} g \text{ (I) } f, g \in \mathfrak{M} \text{ if and only if}$$
$$g_* \leqq f \leqq g^* \text{ (or equivalently } f_* \leqq g \leqq f^*).$$

We denote by $\boldsymbol{C}_{nq}(E)$ the space of the equivalence-classes of $\mathfrak{M}$ by the relation (I). Clearly we may define an order relation of $\boldsymbol{C}_{nq}(E)$, that is, for any $M_1, M_2 \in \boldsymbol{C}_{nq}(E)$ we write $M_1 \leqq M_2$ if and only if $f_* \leqq g_*$ (or $f^* \leqq g^*$) holds,[12] where $f \in M_1$ and $g \in M_2$ respectively.

12) Indeed, as $f, g \in \mathfrak{M}$, $f_* \leqq g_*$ implies $f^* \leqq g^*$ and conversely.

Let $f, g \in \mathfrak{M}$ with $f - g \in N$. Then we have $g_* \leqq f^*$ and $(g_*)^* \leqq f^*$. As g is normal upper semi-continuous, we obtain

$$g \leqq g^* = (g_*)^* \leqq f^*.$$

Similarly we can show that $f - g \in N$ implies $f_* \leqq g$. Hence we may conclude that $f \equiv g$ (N) implies $f \equiv g$ (I) for $f, g \in \mathfrak{M}$. On the other hand, since $f^*(x) = f_*(x)$ holds for every $x \in EA'_f$ for any $f \in \mathfrak{M}$, $f \equiv g$ (I) implies $f \equiv g$ (N). Therefore there exists an one to one mapping from $C_q(E)$ to $C_{nq}(E)$, which satisfies also $\dot{f} \leqq \dot{g}$ if and only if $f \leqq g$ (I) in $\mathfrak{M}$. Since we can see easily that $C_{nq}(E)$ comes to be linear space in virtue of this mapping, we obtain by virtue of Theorem 1

Theorem 2. *The cut extension of $C(E)$ is isomorphic to $C_{nq}(E)$, that is, the space of all normal quasi-continuous functions on E, where two elements f, g are identified if their upper functions (or equivalently lower functions) coincide.*

References

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