103. Relations among Topologies on Riemann Surfaces. II

By Zenjiro KURAMOCHI
Mathematical Institute, Hokkaido University
(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1962)

Proof of Lemma 4. We can suppose without loss of generality that ∂E^i and ∂F^i are regular for the Dirichlet problem. By $E^1 \supset E^2$ $G^{F^i}_{E^2}(z, z_0) - G^{F^i}_{E^1}(z, z_0) \geqq 0$ is clear. Since $G^{F^2}_{E^2}(z, z_0) - G^{F^2}_{E^1}(z, z_0) = 0$ on ∂F^2, by the minimum principle we have $G^{F^2}_{E^2}(z, z_0) - G^{F^2}_{E^1}(z, z_0) \geqq 0$ on ∂F^1. On the other hand, $G^{F^1}_{E^2}(z, z_0) - G^{F^1}_{E^1}(z, z_0) = 0$ on ∂F^1. Next $G^{F^2}_{E^2}(z, z_0) - G^{F^2}_{E^1}(z, z_0) = G^{F^2}_{E^2}(z, z_0) \geqq G^{F^1}_{E^2}(z, z_0) = G^{F^1}_{E^2}(z, z_0) - G^{F^1}_{E^1}(z, z_0)$ on ∂E^1. Thus we have by the maximum principle

$$G^{F^2}_{E^2}(z, z_0) - G^{F^1}_{E^2}(z, z_0) \geqq G^{F^2}_{E^1}(z, z_0) - G^{F^1}_{E^1}(z, z_0). \qquad (2)$$

By definition we have $G^L_K(z, z_0) = G^{L+K}(z, z_0) = G^K_L(z, z_0)$. Put $F^1 = F^1_1$, $F^2 = F^2_1$, $E^1 = \sum_{i=2}^{n} F^1_i$ and $E^2 = \sum_{i=2}^{n} F^2_i$. Then $G^{E^2+F^2}(z, z_0) - G^{E^1+F^2}(z, z_0) = (G^{F^2}_{E^2}(z, z_0) - G^{F^1}_{E^2}(z, z_0)) + (G^{E^2}_{F^1}(z, z_0) - G^{E^1+F^1}(z, z_0)) = (G^{F^2}_{E^2}(z, z_0) - G^{F^1}_{E^2}(z, z_0)) + (G^{E^2}_{F^1}(z, z_0) - G^{E^1}_{F^1}(z, z_0)) \leqq (G^{F^2}(z, z_0) - G^{F^1}(z, z_0)) + (G^{E^2}(z, z_0) - G^{E^1}(z, z_0))$ by (2). In this way proceed, then we have

$$G^{\sum_i F^2_i}(z, z_0) - G^{\sum_i F^1_i}(z, z_0) \leqq \sum_i (G^{F^2_i}(z, z_0) - G^{F^1_i}(z, z_0)). \qquad (3)$$

Lemma 5. *Let D be a simply connected domain and let $L = E[z: 0 \leqq Re\, z \leqq a,\ Im\, z = 0]$ be a segment and let R be a closed set such that $D-L-R$ is simply connected.*

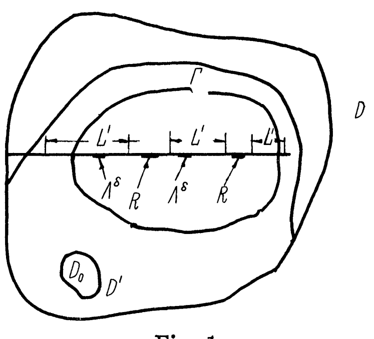


Fig. 1

Let Λ^δ_i be a closed segment on $L-R$ such that $\Lambda^\delta_i = E[z: |z-a_i| < \delta,\ Im\, z = 0]$ and $0 < a_1 < a_2 \cdots < a_n < a$. Put $\Lambda^\delta = \sum \Lambda^\delta_i$. Let D' and Γ be simply connected domains such that $D \supset D' \supset (\Lambda^\delta + R)$, dist $(\partial\Gamma, \Lambda^\delta) > 0$ *for* $\delta < \delta_0$, dist $(\partial\Gamma, \partial D') > 0$ *and $D'-L-R$ is also simply connected. Let D_0 be a compact domain in D' such that* dist $(\Gamma, D_0) > 0$. *Let $w(z, \Lambda^\delta, D-L-R)$ be the harmonic measure of Λ^δ relative to $D-L-R$ and let $G(z, z_0, D')$ be the Green's function of D'. Then for any given positive number ε we can find a constant $\delta(\varepsilon)$ such that*

$$\frac{w(z, \Lambda^\delta, D-L-R)}{G(z, z_0, D')} < \varepsilon \text{ on } \partial\Gamma \text{ for } \delta < \delta(\varepsilon).$$

Let z_0 be a fixed point in D. Map $D-L$ conformally onto $|\xi| < 1$ by $\xi = f(z)$ so that $z_0 \to \xi = 0$. Let L' be a closed subset of $(L-R)$

$\cap D'$ such that L' is contained completely in D' and containing $\partial\Gamma\cap L$. Then $\xi=f(z)$ is analytic on L'. Hence there exist constants N_1 and M_1 such that

$$0<N_1<|f'(z)|<M_1<\infty \text{ in a neighbourhood of } L'. \quad (4)$$

Since dist $(\partial\Gamma, \Lambda^\delta)>0$ implies dist $(\partial\Gamma_\xi, \Lambda^\delta_\xi)>0$, $\varliminf_{\substack{|\xi_1|\to 1\\ \xi_2\in\Lambda^\delta_\xi}} |\arg\xi_1-\arg\xi_2|>0$: $\xi_1\in\partial\Gamma_\xi$, where Γ_ξ and Λ^δ_ξ are the images of Γ and Λ^δ. On the other hand, $w(z, \Lambda^\delta, D-R-L)=w(\xi, \Lambda^\delta_\xi)=\frac{1}{2\pi}\int_{\Lambda^\delta_\xi}\frac{(1-r^2)}{(1-2r\cos(\theta-\varphi)+r^2)}d\varphi: re^{i\theta}=\xi$. Hence

$$w(z, \Lambda^\delta, D-L-R)\leqq\frac{\text{length of } \Lambda^\delta_\xi}{2\pi}\times(1-r^2) \text{ as } z\to L \text{ and } z\in\Gamma. \quad (5)$$

Denote $E[z\in\partial\Gamma: \text{dist}(z, L)<h]$ by $\partial\Gamma^h$. Then by (4) there exist constants δ_3, M_2 and δ_4 such that

$$w(z, \Lambda^\delta, D-L-R)\leqq M_2(\text{length of } \Lambda_\delta)h \text{ for } z\in\partial\Gamma^h,\ \delta<\delta_3,\ h<\delta_4, \quad (6)$$

where $h=\text{dist}(z, L)$).

Map D' onto $|\zeta|<1$ by $\zeta=g(z)$ so that $z_0\to\zeta=0$. Then $g(z)$ is analytic on L' and $g'(z)$ is *continuous in a neighbourhood of L' with respect to z_0*, because D_0 is compact. Hence there exist constants N_3, M_3 and δ_5 such that $0<N_3<g'(z)<M_3$ for $z\in\Gamma$ and dist $(z, L')<\delta_5$. Now $G(z, z_0, D')=\log\frac{1}{|\zeta|}$. Hence there exist constants and N_4 such that

$$G(z, z_0, D')\geqq h\ N_4 \text{ in } \partial\Gamma^{\delta_5} \text{ for } h<\delta_6, \quad (7)$$

because $\frac{\partial}{\partial n}G(\zeta, O, D)=1$ at $|\zeta|=1$. On the other hand,

$$G(z, z_0, D')>N_4>0 \text{ for } z\in(\partial\Gamma-\partial\Gamma^{\delta_5}). \quad (8)$$

Hence by (6), (7) and (8) we can choose $\delta(\varepsilon)$ such that

$$\frac{w(z, \Lambda^\delta, D-L-R)}{G(z, z_0, D')}<\varepsilon \text{ on } \partial\Gamma \text{ for } \delta<\delta(\varepsilon) \text{ and for any } z_0\in D_0.$$

Lemma 6. *Let $D_n(n=1,2,\cdots)$ be a domain such that $D_n\uparrow D$. Let D_0 be a compact domain in D_1. Let $\{p^i_m\}$ $(i=1,2,\ m=1,2,\cdots)$ be a sequence such that $\{p^i_m\}$ determine the same K-Martin's point relative to D_n for every n, in other words, $\lim_m K(z, p^1_m, D_n)=\lim_m K(z, p^2_m, D_n)$, $K(z, p^i_m, D_n)=\frac{G(z, p^i_m, D_n)}{G(p_0, p^i_m, D_n)}$ and p_0 is a fixed point in D_0. Let (z, z_0, D_n) and $G(z, z_0, D)$ be Green's functions of D_n and D respectively. If $\frac{G(p^i_m, z, D)-G(p^i_m, z, D_n)}{G(p^i_m, z, D)}<\varepsilon_n$ for any $z\in D_0$ and $\lim_n \varepsilon_n=0$ $(i=1,2)$, then $\{p^1_m\}$ and $\{p^2_m\}$ determine the same K-Martin's point relative to D.*

In fact, from the above inequality we have

$$\left|\lim_m \frac{G(p_m^i, z, D_n)}{G(p_m^i, p_0, D_n)} - \lim_m \frac{G(p_m^i, z, D)}{G(p_m^i, p_0, D)}\right| < \frac{\varepsilon_n}{(1-\varepsilon_n)} \overline{\lim_m} \frac{G(p_m^i, z, D)}{G(p_m^i, p_0, D)}$$

$$= \frac{\varepsilon_n}{(1-\varepsilon_n)} \overline{\lim_m} K(p_m^i, z, D) < \frac{\varepsilon_n}{(1-\varepsilon_n)} M(D_0) \text{ in } D_0,$$

where $M(D_0)=\sup_{z\in D_0}(\overline{\lim_m} K(p_m^i, z, D))<\infty$. Since $p\{_m^1\}$ and $\{p_m^2\}$ determine the same point, we have by $\frac{G(p_m^i, z, D_n)}{G(p_m^i, p_0, D_n)} = K(p_m^i, z, D_n)$

$$|\lim_m K(p_m^1, z, D) - \lim_m K(p_m^2, z, D)| < \frac{2\varepsilon_n M(D_0)}{1-\varepsilon_n} \text{in } D_0.$$

Let $\varepsilon_n \to 0$. Then $\lim_m K(p_m^1, z, D) = \lim_m K(p_m^2, z, D)$ in D_0, whence $\lim_m K(p_m^1, z, D) = \lim_m K(p_m^2, z, D)$ for $z \in D$. Thus $\{p_m^1\}$ and $\{p_m^2\}$ determine the same K-Martin's point relative to D.

Example 3. *Domain D^*. Let m_n $(n=1, 2, 3, \cdots)$ be a positive number such that*

$$\sum_{n=1}^{\infty} \frac{1}{m_n} \leqq \frac{1}{72\pi}$$

and put $a_n = \frac{6}{2^{n+2}} e^{-m_n}$. *Then* $\log \frac{(6/2^{n+2})}{a^n} = m_n$.

Let $\mathfrak{R}$ be a square, $\tilde{s}_n$, t_n, s_n^1, s_n^2 and s_n^3 be slits and R_n be a rectangle as follows:

$\mathfrak{R}$: $0 < Re\, z < 6$, $0 < Im\, z < 6$.

$\tilde{s}_n$: $Re\, z = 3$, $6 \geqq Im\, z \geqq 4.5 + a_1$ for $n=0$ and

$\tilde{s}_n$: $Re\, z = 3$, $3\left(\frac{1}{2^{n-1}} + \frac{1}{2^n}\right) - a_n \geqq Im\, z \geqq 3\left(\frac{1}{2^{n+1}} + \frac{1}{2^n}\right) + a_{n+1} : n \geqq 1$.

t_n: $Re\, z = 3$, $3\left(\frac{1}{2^{n-1}} + \frac{1}{2^n}\right) + a_n \geqq Im\, z \geqq 3\left(\frac{1}{2^{n-1}} + \frac{1}{2^n}\right) - a_n : n \geqq 1$.

R_n: $\alpha \leqq Re\, z \leqq \alpha + 1$, $\frac{6}{2^n} + \frac{6}{2^{n+4}} \geqq Im\, z \geqq \frac{6}{2^n} - \frac{6}{2^{n+4}}$, where α is 1 or 4 according as n is odd or even.

s_n^1: $0 \leqq Re\, z \leqq 1$, $Im\, z = \frac{6}{2^n}$. s_n^2: $2 \leqq Re\, z \leqq 4$, $Im\, z = \frac{6}{2^n}$.

s_n^3: $5 \leqq Re\, z \leqq 6$, $Im\, z = \frac{6}{2^n}$.

Put $D^* = \mathfrak{R} - \sum_{n=1}^{\infty} (\tilde{s}_n + R_n + s_n^1 + s_n^2 + s_n^3) - \tilde{s}_0$.

Domain ${}_e\mathfrak{D}_m$, $l < m$. Slits Λ_n and domains Δ_0 and Δ_0'. Let Δ_0^i $(i=1,2)$ as follows:

$$\Delta_0^i = E[z : \alpha \leqq Re\, z \leqq \alpha + 1,\ 4 \leqq Im\, z \leqq 5],$$

where $\alpha = 1$ or 4 according as $i=1$ or 2. Put $\Delta_0 = \Delta_0^1 + \Delta_0^2$ and $\Delta_0' = E\left[z : \text{dist}\,(z, \Delta_0) \leqq \frac{1}{2}\right]$.

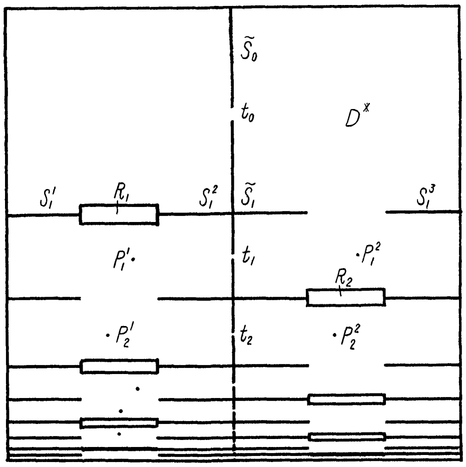


Fig. 2

Let Γ_n be a simply connected domain containing R_n as follows: $\Gamma_n: \alpha-1 \leqq Re\, z \leqq \alpha+1,\ \frac{6}{2^n}-\frac{6}{2^{n+3}} \leqq Im\, z \leqq \frac{6}{2^n}-\frac{6}{2^{n+3}}$ and let Λ_n^1 and Λ_n^2 be segments on $s_n^1+s_n^2$ (for odd n) or on $s_n^2+s_n^3$ (for even n) such that $\Lambda_n^1: \alpha-0.75-\alpha_n \leqq Re\, z \leqq \alpha-0.75+\alpha_n$,

$\Lambda_n^2: \alpha+0.75-\alpha_n \leqq Re\, z \leqq \alpha+0.75+\alpha_n,\ (0<\alpha_n<0.2)$,

where $\alpha=1.5$ or 4.5 according as n is odd or even. Put $\Lambda_n=\Lambda_n^1+\Lambda_n^2$.

Put $D^{s_n}=\mathfrak{R}-s_n^1-R_n-s_n^2$ (for odd n) and $=\mathfrak{R}-s_n^2-R_n-s_n^3$ (for even n). Let $w(z, \Lambda_n, D^{s_n})$ be the harmonic measure of Λ_n relative to D^{s_n}. Let $G(z, z_0, \mathfrak{R})$ be the Green's function of $\mathfrak{R}$. Put $M_n=\max G(z, z_0, \mathfrak{R})$ on $\partial\Gamma_n$ as z_0 varies in Δ_0. Then $M_n<\infty$. Let $G(z, z_0, D^*)$ be the Green's function of $D^*: z_0 \in \Delta_0$. Now D^{s_n} and D^* are simply connected. Hence by Lemma 5 we can find α_n such that

$$M_n w(z, \Lambda_n, D^{s_n}) \leqq \frac{1}{4^n} G(z, z_0, D^*) \text{ on } \partial\Gamma_n \text{ for any } z_0 \in \Delta_0. \tag{8}$$

We suppose that α_n is determined as (8) and Λ_n is defined for every n.

Let $'s_n^1$ and $'s_n^3$ be segments on s_n^1 and s_n^3 such that

$'s_n^1: 0 \leqq Re\, z \leqq 0.75-\alpha_n$ and $'s_n^3=s_n^3$ for odd number n,

$'s_n^1=s_n^1$ and $'s_n^3: 5.25+\alpha_n \leqq Re\, z \leqq 6$ for even number n.

Then $'s_n^1 \subset s_n^1$ and $'s_n^3 \subset s_n^3$.

Let p_n^i ($i=1, 2$ and $n=1, 2, 3, \cdots$) be a sequence such that p_n^i: $c_n^i+\frac{1}{2}\left(\frac{6}{2^n}+\frac{6}{2^{n+1}}\right)i$, where $1<c_n^1<2$ for $i=1$ and $4<c_n^2<5$ for $i=2$.

Put $D_m=\mathfrak{R}-\tilde{s}_0-\sum_1^m('s_n^1+'s_n^3)-\sum_{m+1}^\infty(\tilde{s}_n+R_n+s_n^1+s_n^2+s_n^3)$. Then D_m is simply connected. Map D_m onto $|\zeta|<1$. Then since $\{t_n\}: n>m+2$ is a fundamental sequence determining a prim Ende, the images of $\{p_n^1\}$

and $\{p_n^2\}$ tend to the same point for any c_n^i. Hence we have the following

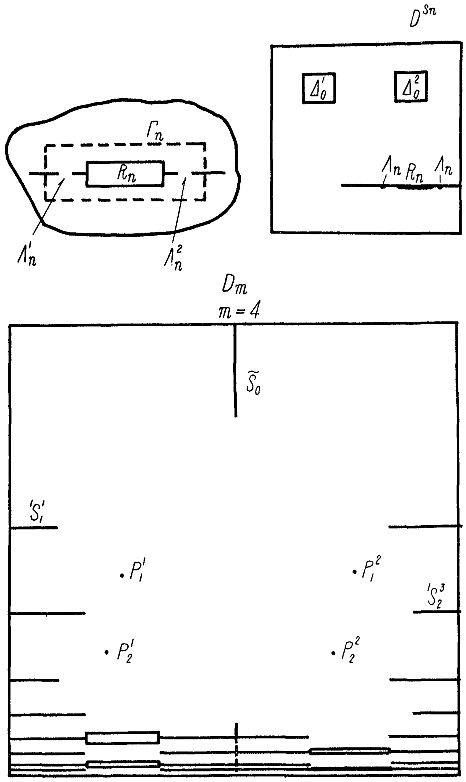


Fig. 3

Proposition 1. *$\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point relative to D_m for any m.*

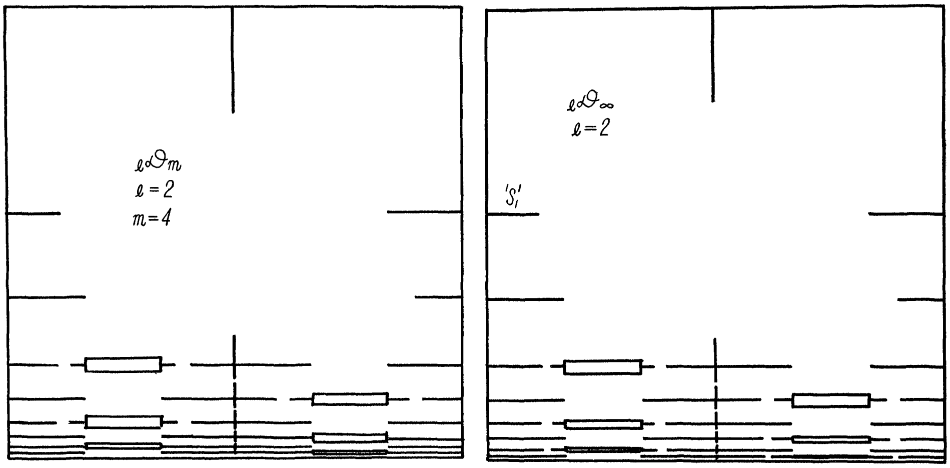
Put ${}_l\mathfrak{D}_m = D_m - \sum_{l+1}^{m}(s_n^1+s_n^2+s_n^3+R_n+\tilde{s}_n-\Lambda_n) = \mathfrak{R}-\tilde{s}_0-\sum_{1}^{l}({}'s_n^1+{}'s_n^3) - \sum_{l+1}^{m}(s_n^1+s_n^2+s_n^3+\tilde{s}_n+R_n-\Lambda_n) - \sum_{m+1}^{\infty}(s_n^1+s_n^2+s_n^3+R_n+\tilde{s}_n)$. Then $D_m - {}_l\mathfrak{D}_m$ is compact in D_m. Hence by Lemma 1 and Proposition 1 we have the following

Proposition 2. *$\{p_n^1\}$ and $\{p_n^2\}$ determine the same K-Martin's point relative to ${}_l\mathfrak{D}_m$, i.e. $\lim_n K^{{}_l\mathfrak{D}_m}(p_n^1, z) = \lim_n K^{{}_l\mathfrak{D}_m}(p_n^2, z)$: $K^{{}_l\mathfrak{D}_m}(p_n^i, z) = \dfrac{G(z, p_n^i, {}_l\mathfrak{D}_m)}{G(p_0, p_n^i, {}_l\mathfrak{D}_m)}$ and p_0 is a fixed point in Δ_0.*

The domain ${}_l\mathfrak{D}_m \uparrow {}_l\mathfrak{D}_\infty = \mathfrak{R}-\tilde{s}_0-\sum_{1}^{l}({}'s_n^1+{}'s_n^3)-\sum_{l+1}^{\infty}(s_n^1+s_n^2+s_n^3+\tilde{s}_n+R_n-\Lambda_n)$ as $m\to\infty$. By $D^{s_n}+\Lambda_n \supset {}_l\mathfrak{D}_m \supset D^*$ for any n we have $w(z, \Lambda_n, {}_l\mathfrak{D}_m) \leqq w(z, \Lambda_n, D^{s_n})$ and $G(z, z_0, {}_l\mathfrak{D}_m) \geqq G(z, z_0, D^*)$. Consider $G(z, z_0, {}_l\mathfrak{D}_\infty)$ and $G(z, z_0, {}_l\mathfrak{D}_m)$ in ${}_l\mathfrak{D}_m$. Then $G(z, z_0, {}_l\mathfrak{D}_\infty) \geqq G(z, z_0, {}_l\mathfrak{D}_m) = 0$ on $\partial {}_l\mathfrak{D}_m$

and $M_n w(z, A_n, {}_l\mathfrak{D}_m) \geqq G(z, z_0, \mathfrak{R}) \geqq G(z, z_0, {}_l\mathfrak{D}_\infty) \geqq G(z, z_0, {}_l\mathfrak{D}_m) = 0$ on $\sum_{m+1}^{\infty} A_n$ for any $z_0 \in \Delta_0$. Hence by the maximum principle

$$\sum_{m+1}^{\infty} M_n w(z, A_n, {}_l\mathfrak{D}_m) + G(z, z_0, {}_l\mathfrak{D}_m) \geqq G(z, z_0, {}_l\mathfrak{D}_\infty)$$
$$\geqq G(z, z_0, {}_l\mathfrak{D}_m) \text{ in } {}_l\mathfrak{D}_m : z_0 \in \Delta_0. \qquad (10)$$



(a) (b)

Fig. 4

By (8) $\frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_m) \geqq \frac{1}{4^n} G(z, z_0, D^*) \geqq M_n w(z, A_n, D^{s_n}) \geqq M_n w(z, A_n, {}_l\mathfrak{D}_m)$ on $\partial \Gamma_n$. On the other hand, $\frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_m) = 0 = M_n w(z, A_n, {}_l\mathfrak{D}_m)$ on $\partial_l \mathfrak{D}_m - \Gamma_n$, whence by the maximum principle $\frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_m) \geqq M_n w(z, A_n, {}_l\mathfrak{D}_m)$ in ${}_l\mathfrak{D}_m - \Gamma_n$. Hence

$$\left(\sum_{m+1}^{\infty} \frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_\infty) \geqq \right) \sum_{m+1}^{\infty} \frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_m) \geqq \sum_{m+1}^{\infty} M_n w(z, A_n, {}_l\mathfrak{D}_m)$$
$$\text{in } {}_l\mathfrak{D}_m - \sum_{m+1}^{\infty} \Gamma_n : z_0 \in \Delta_0. \qquad (11)$$

Thus by (10) and (11) $\sum_{m+1}^{\infty} \frac{1}{4^n} G(z, z_0, {}_l\mathfrak{D}_m) + G(z, z_0, {}_l\mathfrak{D}_m) \geqq G(z, z_0, {}_l\mathfrak{D}_\infty) \geqq G(z, z_0, {}_l\mathfrak{D}_m)$ in ${}_l\mathfrak{D}_m - \sum_{m+1}^{\infty} \Gamma_n$. Now $\{p_n^i\} \in {}_l\mathfrak{D}_m - \sum_{m+1}^{\infty} \Gamma_n$. Put $\varepsilon_m = \sum_{m+1} \frac{1}{4^n}$. Then $\lim_m \varepsilon_m = 0$. Hence $G(p_n^i, z_0, {}_l\mathfrak{D}_\infty) - G(p_n^i, z_0, {}_l\mathfrak{D}_m) < \varepsilon_m G(p_n^i, z_0, {}_l\mathfrak{D}_m) \leqq \varepsilon_m G(p_n^i, z_0, {}_l\mathfrak{D}_\infty)$. Hence by Proposition 2 and by Lemma 6 we have the following proposition which is given in the following paper.