99. On the Normalizer of an f-Ring

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1. *Introduction.* D. G. Johnson has studied the structure of f-rings, that is, lattice ordered rings R which satisfy the following condition:

For $a, b, c \in R$, if $c \geq 0$ and $a \frown b = 0$, then $ca \frown b = ac \frown b = 0$.

In [4] he shows that an f-ring without nilpotent elements can be embedded as a sub-f-ring of an f-ring with an identity. In [7] Henriksen and Isbell have shown that f-ring without left or right annihilators can also be embedded in an f-ring with identity. In this note we intend to prove following theorem which improves these results.

THEOREM 3.1. *If R is an f-ring without non-zero right or left annihilators, then the normalizer of R is an f-ring which is, of course, without non-zero right or left annihilators.*

The methods employed in this note are quite different from those used by either Johnson or Henriksen and Isbell.

Recall that a *left annihilator* of ring R is an element $a \in R$ with the property: $aR = 0$. A right annihilator is defined analogously.

R. E. Johnson in [5] discusses the normalizer $N^l(R)$ of a ring R without non-zero left annihilators. This ring is composed of those mappings φ of R into R which satisfy the following conditions:

(N 1) $\varphi(a+b) = \varphi(a) + \varphi(b)$ for $a, b, \in R$.

(N 2) $\varphi(ab) = \varphi(a)b$ for $a, b, \in R$.

(N 3) For each $a \in R$, there is an element $(a)\rho_\varphi \in R$ such that $a\varphi(x) = (a)\rho_\varphi x$ for every $x \in R$.

The conditions (N 1) and (N 2) indicate that $N^l(R)$ is a subset of the ring $E^l(R)$ of R-endomorphisms of R considered as a left R-module. The set $N^l(R)$ is closed with respect to the ring operations of $E^l(R)$ and hence is a sub-ring of $E^l(R)$. The set $N^l(R)$, endowed with the operations of $E^l(R)$, we shall call the *normalizer of R*.

Since 0 is the only left annihilator of R in R, the mapping $a \rightarrow T_a$ is an isomorphism of R into $N^l(R)$ provided T_a is defined to be the mapping $T_a : x \rightarrow ax$.

In [5], R. E. Johnson shows that if the ring R is an ideal of a ring S and if the only left annihilator of R in S is zero, then S is isomorphic to a subring of $N^l(R)$. In addition, one easily verifies that the isomorphism can be chosen so as to be the identity mapping of $R \subseteq S$ onto $R \subseteq \boldsymbol{N}^l(R)$, that is the isomorphism can be chosen to be an isomorphism over R.

It is also clear that $\boldsymbol{N}^l(R)$ contains the identity mapping of R onto R, and R is embedded as an ideal of $\boldsymbol{N}^l(R)$.

The author in [2] has studied the normalizers of a special class of f-rings from a different point of view. Normalizers for Banach algebras have been defined and studied by Choda and Nakamura [3] and by Wang [6].

2. *The right normalizer.* In the Introduction the left normalizer $\boldsymbol{N}^l(R)$ is defined. If R has no right annihilators except zero, then a right normalizer $\boldsymbol{N}^r(R)$ can also be defined with properties symmetric to those of $\boldsymbol{N}^l(R)$. In order to avoid notational confusion the elements of $\boldsymbol{N}^r(R)$ are written as operators on the right and the elements of $\boldsymbol{N}^l(R)$ as operators on the left.

In this section we show that there is a natural isomorphism between $\boldsymbol{N}^r(R)$ and $\boldsymbol{N}^l(R)$ provided the only annihilator of R on either side in R is 0. For the remainder of this Section assume that R has neither right nor left non-zero annihilators.

From (N 3) it follows that for $\varphi \in \boldsymbol{N}^l(R)$ there is a mapping $(\cdot)\rho_\varphi$ of R into R with the property:

$$(a)\rho_\varphi x = a\varphi(x) \tag{2.1}$$

for each $a \in R$ and each $x \in R$.

LEMMA 2.1. The mapping $(\cdot)\rho_\varphi$ is an element of $\boldsymbol{N}^r(R)$.

PROOF. If $a, b \in R$, then

$$\begin{aligned}(a+b)\rho_\varphi x &= (a+b)\varphi(x) = a\varphi(x) + b\varphi(x) = (a)\rho_\varphi x + (b)\rho_\varphi x \\ &= [(a)\rho_\varphi + (b)\rho_\varphi]x\end{aligned}$$

for each $x \in R$. Thus

$$[(a+b)\rho_\varphi - (a)\rho_\varphi (b)\rho_\varphi]R = 0,$$

and so the mapping $(\cdot)\rho_\varphi$ is additive.

Since

$$a(b)\rho_\varphi x = ab\varphi(x) = (ab)\rho_\varphi x$$

for all $a, b, x \in R$, it follows that

$$a(b)\rho_\varphi = (ab)\rho_\varphi,$$

and so the right-hand analogue of (N 2) is satisfied by $(\cdot)\rho_\varphi$.

The right-hand analogue of (N 3) follows directly for $(\cdot)\rho_\varphi$ from Equation (2.1).

THEOREN 2.1. *The mapping $\varphi \to (\cdot)\rho_\varphi$ is an isomorphism of $\boldsymbol{N}^l(R)$ onto $\boldsymbol{N}^r(R)$.*

PROOF. If φ, $\psi \in \boldsymbol{N}^l(R)$, then

$$\begin{aligned}(a)\rho_{\varphi+\psi} x &= a[\varphi+\psi](x) = a\varphi(x) + a\psi(x) = (a)\rho_\varphi x + (a)\rho_\psi x \\ &= (a)[\rho_\varphi + \rho_\psi]x\end{aligned}$$

for all $a, x \in R$. Since R has no non-zero right annihilators,

$$\rho_{\varphi+\psi}=\rho_\varphi+\rho_\psi.$$

In addition,

$$(a)\rho_{\varphi\psi}x=a[\varphi\psi](x)=a\varphi(\psi(x))=(a)\rho_\varphi\psi(x)=((a)\rho_\varphi)\rho_\psi x$$

for all $x, a\in R$. Thus

$$\rho_{\varphi\psi}=\rho_\varphi\rho_\psi$$

for $\varphi, \psi\in \boldsymbol{N}^l(R)$. Hence $\varphi\to(\cdot)\rho_\varphi$ is a homomorphism of the ring $\boldsymbol{N}^l(R)$ into $\boldsymbol{N}^r(R)$.

If we observe that $\boldsymbol{N}^r(R)$ contains a mapping if and only if it satisfies the right-hand analogue of (N 3), then it is clear from the right-hand analogue of Lemma 2.1 that $\varphi\to(\cdot)\rho_\varphi$ maps $\boldsymbol{N}^l(R)$ onto $\boldsymbol{N}^r(R)$.

To show $\varphi\to(\cdot)\rho_\varphi$ is one to one, note that if $(a)\rho_\varphi=(a)\rho_\psi$ for all $a\in R$, then

$$a\varphi(x)=(a)\rho_\varphi x=(a)\rho_\psi x=a\psi(x)$$

for all $a, x\in R$. Thus

$$\varphi(x)=\psi(x)$$

for all $x\in R$, and hence $\varphi=\psi$. Therefore the theorem is proved.

COROLLARY 2.1. If R is commutative, then

$$\boldsymbol{N}^l(R)=E^l(R) \text{ and } \boldsymbol{N}^r(R)=E^r(R).$$

In addition the mapping $\varphi\to(\cdot)\rho_\varphi$ is the identity isomorphism, that is $\varphi(x)=(x)\rho_\varphi$ for all $x\in R$.

PROOF. If R is commutative, then

$$(a)\rho_\varphi x=a\varphi(x)=\varphi(x)a=\varphi(xa)=\varphi(a)x \text{ for all } x, a\in R.$$

Thus (N 3) is valid for all mappings in $E^l(R)$ and $\boldsymbol{N}^l(R)=E^l(R)$. Analogously, $\boldsymbol{N}^r(R)=E^r(R)$.

Since for all $x, a\in R$,

$$(a)\rho_\varphi x=\varphi(a)x,$$

it follows that $(\cdot)\rho_\varphi$ and $\varphi(\cdot)$ constitute the same mapping.

3. *The normalizer of an f-ring.* In this section let R stand for an f-ring without non-zero left or right annihilators.

In order to show that Theorem 3.1 is a true generalization of D. G. Johnson's result we consider the following example.

EXAMPLE 3.1. Let S be the algebra over the rational numbers generated by the elements e, z where $e^2=e$, $ez=ze=z$ and $z^2=0$.

Lexicographically order S with e dominent: thus $\alpha e+\beta z\geq 0$ provided either $\alpha>0$ or $\alpha=0$ and $\beta\geq 0$. It is clear that S is an ordered group with respect to the relation "$\leq$".

If $\alpha e+\beta z\geq 0$ and $\gamma e+\delta z\geq 0$

then it can be easily verified that

$$(\alpha e+\beta z)(\gamma e+\delta z)=\alpha\gamma e+(\alpha\delta+\beta\gamma)z\geq 0.$$

Thus S is an ordered ring. Since $a\frown b=0$ in an ordered ring implies either $a=0$ or $b=0$, it can easily be verified that S is an f-ring. By construction S contains a nilpotent element z, while

$(\alpha e+\beta z)S=0$ implies that $\alpha=\beta=0$. Therefore, since S is commutative, it has no non-zero left or right annihilators. Thus S constitutes an f-ring with nilpotent elements which has neither left nor right non-zero annihilators.

If T is an f-ring with no nilpotent elements but without an identity, then the direct sum $T\oplus S$ is a ring with nilpotent elements which does not contain an identity and does not contain non-zero left or right annihilators. $T\oplus S$, however, can be embeded in an f-ring with identity.

In order to prove Theorem 3.1 we prove the following sequence of lemmas.

LEMMA 3.1. The ring $\boldsymbol{N}^l(R)$ is a partially ordered ring with respect to the relation: $\varphi\geq\psi$ if and only if $\varphi(x)\geq\psi(x)$ for all $x\geq 0$ in R.

PROOF. The following statements are easily verifiable:

(1) $\varphi\geq\psi\Rightarrow\varphi+\xi\geq\psi+\xi$ for $\varphi,\psi,\xi\in\boldsymbol{N}^l(R)$.

(2) $\varphi\geq\psi$ and $\xi\geq 0\Rightarrow\varphi\xi\geq\psi\xi$ and $\xi\varphi\geq\xi\psi$ for $\varphi,\psi,\xi\in\boldsymbol{N}^l(R)$.

(3) $\varphi\geq\psi$ for $\varphi\in\boldsymbol{N}^l(R)$.

(4) $\varphi\geq\psi$ and $\psi\geq\xi\Rightarrow\varphi\geq\xi$ for $\varphi,\psi,\xi\in\boldsymbol{N}^l(R)$.

To show $\boldsymbol{N}^l(R)$ is a p.o. ring we need only show:

$$\varphi\geq\psi \text{ and } \psi\geq\varphi\Rightarrow\varphi=\psi.$$

If $\varphi(x)-\psi(x)\geq 0$ and $\psi(x)-\varphi(x)\geq 0$ for all $x\geq 0$ in R, then $\varphi(x)=\psi(x)$ for all $x\geq 0$ in R. Since every element $x\in R$ can be written in the form $x=x^+-x^-$ where $x^+\geq 0$ and $x^-\geq 0$ it follows that $\varphi(x)=\psi(x)$ for all $x\in R$, and $\varphi=\psi$.

REMARK 3.1. It is clear that if R is an f-ring, then with respect to the relation, $\rho_\varphi\geq\rho_\psi$ if and only if $(x)\rho_\varphi\geq(x)\rho_\psi$ for all $0\leq x\in R$, $\boldsymbol{N}^r(R)$ is a partially ordered ring. In addition the mapping $\varphi\to(\cdot)\rho_\varphi$ is an isomorphism of the p.o. ring $\boldsymbol{N}^l(R)$ onto the p.o. ring $\boldsymbol{N}^r(R)$.

Let $\varphi^+(x)=[\varphi(x^+)]^+-[\varphi(x^-)]^+$ for each $x\in R$. Then the following lemma is valid.

LEMMA 3.2. For $x\in R$, $0\leq a\in R$ and $\varphi\in\boldsymbol{N}^l(R)$,

$$a\varphi^+(x)=[(a)\rho_\varphi]^+x.$$

Thus φ^+ satisfies condition (N 3).

PROOF. Since by [4, Theorem 3.15] and Equation (2.1),

$$a\varphi^+(x)=[a\varphi(x^+)]^+-[a\varphi(x^-)]^+$$

it follows that

$$a\varphi^+(x)=[(a)\rho_\varphi]^+(x^+-x^-)=[(a)\rho_\varphi]^+x.$$

LEMMA 3.3. The partially ordered ring $\boldsymbol{N}^l(R)$ is a lattice-ordered ring in which φ^+ is the least upper bound of φ and 0 in $\boldsymbol{N}^l(R)$.

PROOF. First show that φ^+ is an element of $\boldsymbol{N}^l(R)$: If $x,y\in R$, $0\leq a\in R$ and $\varphi\in\boldsymbol{N}^l(R)$, then by Lemma 3.2

$$a\varphi^+(x+y)=[(a)\rho_\varphi]^+(x+y),$$

and

$$a[\varphi^+(x)+\varphi^+(y)]=[(a)\rho_\varphi]^+(x+y).$$

Thus

$$R[\varphi^+(x+y)-\{\varphi^+(x)+\varphi^+(y)\}]=0$$

and hence φ^+ satisfies (N 1).

Since it has already been established that φ^+ satisfies (N 3) and since (N 2) is a direct consequence of Lemma 3.2, it follows that $\varphi^+\in \boldsymbol{N}^l(R)$.

It remains only to show that φ^+ is the least upper bound of the elements φ and 0 in $\boldsymbol{N}^l(R)$: Indeed, it is clear that $\varphi^+\geq 0$ and $\varphi^+\geq\varphi$. Suppose $\psi\geq 0$ and $\psi\geq\varphi$. Then $\psi(x)\geq\varphi(x)$ and $\psi(x)\geq 0$ for all $x\geq 0$. Thus

$$\psi(x)=[\psi(x)]^+=\psi^+(x)\geq(\varphi(x))^+=\varphi^+(x)$$

for all $x\geq 0$. Therefore $\psi\geq\varphi^+$ and hence φ^+ is the least upper bound of φ and 0 in $\boldsymbol{N}^l(R)$. Therefore $\boldsymbol{N}^l(R)$ is a lattice-ordered ring.

COROLLARY 3.2. *$\boldsymbol{N}^r(R)$ is a lattice ordered ring with respect to the order relation:*

$\rho_\varphi\geq\rho_\psi$ *if and only if* $(x)\rho_\varphi\geq(x)\rho_\psi$ *for all* $x\geq 0$ *in* R.

LEMMA 3.4. *$\boldsymbol{N}^l(R)$ is an f-ring.*

PROOF. Since $\boldsymbol{N}^l(R)$ has no non-zero right or left annihilators, we can show, using [1, Theorem 14], that $\boldsymbol{N}^l(R)$ is an f-ring if we can show that from $0\leq\xi\in R$, $\varphi\in R$, and $\psi\in R$ it follows that (i) $\xi(\varphi\frown\psi)=\xi\varphi\frown\xi\psi$, (ii) $(\varphi\frown\psi)\xi=\varphi\xi\frown\psi\xi$, (iii) $\xi(\varphi\smile\psi)=\xi\varphi\smile\xi\psi$, and (iv) $(\varphi\smile\psi)\xi=\varphi\xi\smile\psi\xi$.

To establish (i) note that for $x\geq 0$

$$(\varphi\frown\psi)(x)=[\varphi-(\varphi-\psi)^+](x)=\varphi(x)-(\varphi(x)-\psi(x))^+$$
$$=\varphi(x)\frown\psi(x).$$

For $x\geq 0$ and $a\geq 0$,

$$a\xi(\varphi\frown\psi)(x)=a\xi(\varphi(x)\frown\psi(x))=(a)\rho_\xi(\varphi(x)\frown\psi(x))$$
$$=(a)\rho_\xi\varphi(x)\frown(a)\rho_\xi\psi(x)$$

because $\xi\geq 0$ and hence $\rho_\xi\geq 0$ by Remark 3.1. Thus

$$a\xi(\varphi\frown\psi)(x)=a\xi\varphi(x)\frown a\xi\psi(x)=a(\xi\varphi\frown\xi\psi)(x)$$

for all $a\geq 0$ and $x\geq 0$. This is sufficient to yield the result:

$$\xi(\varphi\frown\psi)(x)=(\xi\varphi\frown\xi\psi)(x)$$

for all $x\geq 0$. Therefore since $x=x^+-x^-$ and for any $\upsilon\in\boldsymbol{N}^l(R)$, $\upsilon(x)=\upsilon(x^+)-\upsilon(x^-)$, it follows that

$$\xi(\varphi\frown\psi)=\xi\varphi\frown\xi\psi.$$

A similar argument can be used to verify (iii). To verify (ii) and (iv) use the fact that for $0\leq x\in R$ and $\varphi,\psi\in\boldsymbol{N}^l(R)$,

$$(\varphi\smile\psi)(x)=\varphi(x)\smile\psi(x).$$

Therefore $\boldsymbol{N}^l(R)$ is an f-ring.

Thus the validity of Theorem 3.1 is established.

We conclude with the following theorem.

THEOREM 3.2. *If R is an ordered ring without non-zero right or left annihilators, then $\boldsymbol{N}^r(R)$ is also ordered.*

PROOF. Suppose $\boldsymbol{N}^l(R)$ is not ordered, that is suppose that for some $\varphi\in\boldsymbol{N}^l(R)$, there is an $a\geq 0$ and a $d\geq 0$ both in R such that

$$\varphi^+(a)\neq 0 \text{ and } \varphi^-(d)\neq 0.$$

Then

$$\varphi^+(a)=[\varphi(a)]^+=\varphi(a),$$

and

$$\varphi^-(d)=[\varphi(d)]^-=-\varphi(d).$$

If $a\leq$d, then $\varphi^+(d)\geq\varphi^+(a)>0$. Thus

$$\varphi^+(d)=[\varphi(d)]^+=\varphi(d)>0 \text{ while } \varphi^-(d)=-\varphi(d)>0.$$

Since $\varphi(d)$ cannot be both positive and negative a cannot be less than or equal to d. On the other hand if $d<a$, then

$$0<\varphi^-(d)\leq\varphi^-(a)=-\varphi(a),$$

and so in this case too we are lead to a situation where $\varphi(a)$ is both positive and negative. Thus the assumption that $\boldsymbol{N}^l(R)$ is not ordered is inadmissable.

References

[1] G. Birkhoff and R. S. Pierce: Lattice ordered rings, An. Acad. Brasil Ci., 41-69 (1956).

[2] B. Brainerd: A construction for the normalizer of a ring with local unit with applications to the theory L-algebras, Trans. Amer. Math. Soc., **97**, 237-253 (1960).

[3] H. Choda and M. Nakamura: Multipliers of Banach algebras, Proc. Japan Acad., Acad., **38**, 93-94 (1962).

[4] D. G. Johnson: A structure theory for a class of lattice-ordered rings, Acta Math., **104**, 163-215 (1960).

[5] R. E. Johnson: The imbedding of a ring as an ideal in another ring, Duke Math. Jour., **20**, 569-573 (1953).

[6] J. K. Wang: Multipliers of commutative Banach algebras, Pacific Jour. Math., **11**, 1131-1149 (1961).

[7] M. Henriksen and Isbell: Lattice-ordered rings and function rings, To appear Pacific Jour. Math.