97. On a Product of Summability Methods

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1. The present note is a continuation of a previous paper by the author [3]. O. Szász [13, 14] discussed the following problem concerning the product of two summability methods for sequences: If a sequence $\{s_n\}$ is summable by a regular T_1 method then is the T_2 transform of $\{s_n\}$, where T_2 is a regular sequence-to-sequence method, also summable by the T_1 method to the same sum as before? In what follows we denote $T_1 \cdot T_2$ as the iteration product of these two methods, that is the T_1 transform of the T_2 transform of a sequence. He answered this problem in the affirmative in the several cases. He also gave an example of two regular methods, where T_1 does not imply $T_1 \cdot T_2$. Here we denote "method A implies method B", when any sequence summable A is summable B to the same sum. T. Pati [5], C. T. Rajagopal [7], M. R. Parameswaran [6], M. S. Ramanujan [11, 12], D. Borwein [1] and the author [3] also discussed this problem. M. S. Ramanujan [11] proved the following

Theorem 1. *For a bounded sequence the Abel method A implies the $A\cdot(H^*, \psi)$ method. Here we denote by (H^*, ψ) the regular quasi-Hausdorff method. In the special case when the (H^*, ψ) method gives the circle method of summability (γ, r), the Abel method implies the $A\cdot(\gamma, r)$ method irrespective of whether $\{s_n\}$ is bounded or not.*

The latter part of this theorem was at first established by O. Szász [14]. See for the definition of the quasi-Hausdorff method of summability G. H. Hardy [2] and M. S. Ramanujan [8, 9, 10].

On the other hand M. S. Ramanujan [10] introduced a new method of summability (S^*, ψ) by a modification of the quasi-Hausdorff method. The (S^*, ψ) means of a sequence $\{s_n\}$ are defined by the transformation

(1) $$s_n^* = \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} s_\nu \int_0^1 (1-t)^\nu t^{n+1} d\psi(t) \quad (n=0, 1, 2, \cdots),$$

where $\psi(t)$ is a function of bounded variation in the closed interval $[0, 1]$. This method is regular if, and only if,

(2) $$\psi(1) = \psi(1-0)$$

and

(3) $$\int_{+0}^1 d\psi(t) = 1. \quad \text{(See [10].)}$$

In the special case when, for a given α $(0<\alpha<1)$,

$$\psi(t)=0 \quad \text{for} \quad 0 \leq t < 1-\alpha$$
$$=1 \quad \text{for} \quad 1-\alpha \leq t \leq 1,$$

we have the S_α method of W. Meyer-König [4] and P. Vermes [15]. Concerning the (S^*, ψ) method M. S. Ramanujan [12] proved further the following

Theorem 2. *If $\{s_n\}$ satisfies the following condition: For every t in $0<t<1$, there exists a function $F(x)$ finite for every x in $0<x<1$ such that*

$$\frac{t}{1-xt}\sum_{\nu=0}^{\infty}|s_\nu|\left(\frac{1-t}{1-xt}\right)^\nu \leq F(x), \quad (0<x<1).$$

Then the Abel method implies the $A\cdot(S^*, \psi)$ method, where the (S^*, ψ) method is assumed to be regular.

D. Borwein [1] studied the logarithmic method L. When a sequence $\{s_n\}$ is given we define the L method as follows: If

$$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

tends to a finite limite s as $x \to 1$ in the open interval $(0, 1)$, we say that $\{s_n\}$ is L-summable to s. It is well known that the Abel method implies the L method. (See G. H. Hardy [2].) Concerning this method D. Borwein [1] proved the following

Theorem 3. *If (H, ψ) is a regular Hausdorff method, then the L method implies the $L\cdot(H, \psi)$ method.*

See for the definition of the Hausdorff method of summability G. H. Hardy [2]. The author [3] proved the following

Theorem 4. *If (H^*, ψ) is a regular quasi-Hausdorff method which satisfies the condition*

(4) $$\int_0^\sigma \log t\,|d\psi(t)| \text{ is finite for a positive } \sigma,$$

then the L method implies the $L\cdot(H^*, \psi)$ method for a bounded sequence. In the special case when the (H^*, ψ) method gives the circle method, the L method implies the $L\cdot(\gamma, r)$ method irrespective of whether $\{s_n\}$ is bounded or not.

Here we prove the following

Theorem 5. *If (S^*, ψ) is a regular method which satisfies the condition (4), then the L method implies the $L\cdot(S^*, \psi)$ method for a bounded sequence.*

2. Proof. For the proof we use the method of M. S. Ramanujan [12]. Since the (S^*, ψ) transforms of $\{s_n\}$ are given by (1) we have

(5) $$\sum_{n=0}^{\infty}\frac{s_n^*}{n+1}x^{n+1}=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu=0}^{\infty}\int_0^1\binom{n+\nu}{\nu}(1-t)^\nu t^{n+1}s_\nu d\psi(t)$$

provided the right-hand member exists. To prove this existence we consider the right-hand member with s_ν replaced by $|s_\nu|$ and $\psi(t)$, supposed to be monotonic increasing (as is permissible). The right-hand member with these changes, is

$$\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu=0}^{\infty}\int_0^1\binom{n+\nu}{\nu}(1-t)^\nu t^{n+1}|s_\nu|\,d\psi(t)$$

$$=\int_0^1\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\binom{n+\nu}{\nu}(1-t)^\nu t^{n+1}\frac{x^{n+1}}{n+1}|s_\nu|\,d\psi(t)$$

$$=\int_0^1\sum_{\nu=0}^{\infty}(1-t)^\nu|s_\nu|\,t\sum_{n=0}^{\infty}\binom{n+\nu}{\nu}\frac{x^{n+1}t^n}{n+1}d\psi(t)$$

every inversion of operations being justified by the fact that we have only positive integrands or terms. Since

$$\sum_{n=0}^{\infty}\binom{n+\nu}{\nu}\frac{x^{n+1}t^n}{n+1}=\int_0^x\frac{du}{(1-ut)^{\nu+1}}$$

$$=\begin{cases}\dfrac{1}{t}\log\dfrac{1}{1-xt} & \text{for } \nu=0\\[2mm] \dfrac{1}{t\nu}\left\{\dfrac{1}{(1-xt)^\nu}-1\right\} & \text{for } \nu\geq 1,\end{cases}$$

the last integral is

$$\int_0^1\left[|s_0|\log\frac{1}{1-xt}+\sum_{\nu=1}^{\infty}(1-t)^\nu|s_\nu|\frac{1}{\nu}\left\{\frac{1}{(1-xt)^\nu}-1\right\}\right]d\psi(t).$$

Here we see easily

$$\log\frac{1}{1-xt}\leq\log\frac{1}{1-x},$$

and further from $|s_\nu|\leq M$ we see

$$\sum_{\nu=1}^{\infty}(1-t)^\nu|s_\nu|\frac{1}{\nu}\left\{\frac{1}{(1-xt)^\nu}-1\right\}\leq M\log\frac{1}{1-x},$$

for $0\leq t\leq 1$ and $0<x<1$. Hence the last integral is finite for $0<x<1$ from (3). Therefore we get, from (5),

$$(6)\qquad \sum_{n=0}^{\infty}\frac{s_n^*}{n+1}x^{n+1}$$

$$=\int_0^1\left[s_0\log\frac{1}{1-xt}+\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}(1-t)^\nu\left\{\frac{1}{(1-xt)^\nu}-1\right\}\right]d\psi(t)$$

$$=\int_0^1 s_0\log\frac{1}{1-xt}d\psi(t)+\int_0^1\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\left(\frac{1-t}{1-xt}\right)^\nu d\psi(t)-$$

$$-\int_0^1\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}(1-t)^\nu d\psi(t).$$

Substituting $x=1-\dfrac{1}{y}$, we have

$$\frac{-1}{\log(1-x)}\int_0^1 s_0\log\frac{1}{1-xt}d\psi(t)$$

$$=s_0\int_0^1\frac{\log y-\log(y-yt+t)}{\log y}d\psi(t)$$

$$=s_0 J, \text{ say.}$$

Since

$$\overline{\lim_{y\to\infty}}\,|J|\le\overline{\lim_{y\to\infty}}\left\{1-\frac{\log(y-y\sigma+\sigma)}{\log y}\right\}\int_0^\sigma|d\psi(t)|+\int_\sigma^1|d\psi(t)|,$$

we have

$$\overline{\lim_{y\to\infty}}\,|J|\le\int_\sigma^1|d\psi(t)|$$

for $0<\sigma<1$. Since, by (2),

$$\int_\sigma^1|d\psi(t)|\to 0 \quad \text{as} \quad \sigma\to 1$$

in the open interval $(0,1)$, we have $\overline{\lim}_{y\to\infty}|J|=0$. Next we put

$$f(x)=\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}x^\nu=\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}\left(1-\frac{1}{y}\right)^\nu=g(y).$$

If $\{s_n\}$ is L-summable to s, then

$$\lim_{y\to\infty}\frac{g(y)}{\log y}=s$$

from the translativity of the L method. (See [1].) Since

$$\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}\left(\frac{1-t}{1-xt}\right)^\nu=\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}\left(1-\frac{t}{y-yt+t}\right)^\nu$$
$$=g\left(\frac{y-yt+t}{t}\right)=g\left(1+\frac{y(1-t)}{t}\right)$$
$$=s\log\left(1+\frac{y(1-t)}{t}\right)+o\left(\log\left(1+\frac{y(1-t)}{t}\right)\right), \text{ as } y\to\infty,$$

we have

$$(7)\qquad \frac{-1}{\log(1-x)}\int_0^1\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}\left(\frac{1-t}{1-xt}\right)^\nu d\psi(t)$$
$$=s\int_0^1\frac{\log\left(1+\frac{y(1-t)}{t}\right)}{\log y}d\psi(t)+$$
$$+o\left(\int_0^1\frac{\log\left(1+\frac{y(1-y)}{t}\right)}{\log y}d\psi(t)\right).$$

On the other hand from (3) and (4) we have

$$\int_0^1\frac{\log\left(1+\frac{y(1-t)}{t}\right)}{\log y}d\psi(t)$$
$$=\int_0^1\frac{\log(t+y(1-t))}{\log y}d\psi(t)-\int_0^1\frac{\log t}{\log y}d\psi(t)$$
$$=1+o(1), \text{ as } y\to\infty,$$

similarly as the estimation of J. Hence from (7)

$$\frac{-1}{\log(1-x)}\int_0^1\sum_{\nu=1}^\infty\frac{s_\nu}{\nu}\left(\frac{1-t}{1-xt}\right)^\nu d\psi(t)\to s$$

as $x \to 1$ in the open interval $(0, 1)$.

Finally we have

$$\frac{-1}{\log(1-x)}\int_0^1 \sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}(1-t)^\nu d\psi(t)$$
$$=\frac{1}{\log y}\int_0^1 f(1-t)d\psi(t)=K, \text{ say.}$$

Since $|s_n| \leq M$ or

$$|f(1-t)| \leq M\sum_{\nu=1}^{\infty}\frac{(1-t)^\nu}{\nu}=-M\log t$$

for $0 < t \leq 1$, we have

$$|K| \leq \frac{-M}{\log y}\int_0^1 \log t\,|d\psi(t)|=o(1) \quad \text{as } y \to \infty$$

from the condition (4).

Collecting above estimations

$$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n^*}{n+1}x^{n+1}$$

tends to s as $x \to 1$ in the open interval $(0, 1)$, whence the proof is complete.

3. **Remark.** In the special case when, for a given α $(0 < \alpha < 1)$,

$$\psi(t)=0 \quad \text{for} \quad 0 \leq t < 1-\alpha$$
$$=1 \quad \text{for} \quad 1-\alpha \leq t \leq 1$$

which satisfies all the conditions of our theorem, (5) and (6) become respectively

$$(5') \qquad \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\sum_{\nu=0}^{\infty}\binom{n+\nu}{\nu}\alpha^\nu(1-\alpha)^{n+1}s_\nu$$

and

$$(6') \qquad s_0\log\frac{1}{1-x(1-\alpha)}+\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\alpha^\nu\left\{\frac{1}{(1-x(1-\alpha))^\nu}-1\right\}.$$

Then we get the equality $(5')=(6')$ irrespective of whether $\{s_n\}$ is bounded or not, since $(6')$ converges absolutely in $0 \leq x < 1$. Therefore we have the following

Corollary. *The L method implies the $L\cdot S_\alpha$ method for $0 < \alpha < 1$.*

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