146. Homotopy Groups with Coefficients and a Generalization of Dold-Thom's Isomorphism Theorem. II

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This paper is a continuation of the previous paper I. Notations and definitions of the paper I will be used without any comment.

3. **Infinite symmetric products.** Let X be a Hausdorff space with base point o. The q-fold symmetric product $SP^q(X)$ is a space obtained from the topological product $X^q=X\times\cdots\times X$ (q-fold) by identifying all points which differ only in the order of the components. The image of $(x_1,\cdots,x_q)\in X^q$ by the identification map is denoted by $[x_1,\cdots,x_q]\in SP^q(X)$. We define the inclusion map i_q: $SP^q(X)\to SP^{q+1}(X)$ by $i_q[x_1,\cdots,x_q]=[o,x_1,\cdots,x_q]$. The infinite symmetric product of X with respect to the base point o is the inductive limit of the sequence $X=SP^1(X)\xrightarrow{i_1}SP^2(X)\xrightarrow{i_2}\cdots$ and is denoted by $SP(X,o)$. $SP(X,o)$ is a Hausdorff space (cf. [1], p. 254).

A map $f:(X,o)\to(X',o')$ induces a map $f^{\Sigma q}: SP^q(X)\to SP^q(X')$ defined by $f^{\Sigma q}[x_1,\cdots,x_q]=[f(x_1),\cdots,f(x_q)]$. Clearly, $f^{\Sigma q}$ is compatible with the inclusion i_q. Then a map $f^{\Sigma}: SP(X,o)\to SP(X',o')$ can be defined by $f^{\Sigma}|SP^q(X)=f^{\Sigma q}$. Obviously, if f and g are homotopic, then f^{Σ} and g^{Σ} are homotopic. Hence the homotopy type of $SP(X,o)$ depends only on that of (X,o). It is easily verified that if A is closed (open) in X and $i: A\to X$ is the inclusion, then the induced maps $i^{\Sigma q}: SP^q(A)\to SP^q(X)$ and $i^{\Sigma}: SP(A,o)\to SP(X,o)$ are homeomorphisms into.

An addition $\mu: SP(X,o)\times SP(X,o)\to SP(X,o)$ can be defined by $\mu([x_1,\cdots,x_q],[y_1,\cdots,y_r])=[x_1,\cdots,x_q,y_1,\cdots,y_r]$. $SP(X,o)$ is a free abelian semi-group over X with o as the unit element with respect to the addition μ. μ is continuous on any subset of $SP^q(X)\times SP^r(X)$ and on any compact subset of $SP(X,o)\times SP(X,o)$.

Now let X be a CW-complex, $A\ni o$ a connected subcomplex of X, X/A a space obtained from X by contracting A to a point $\bar{o}$ (the base point of X/A), and let $p:(X,o)\to(X/A,\bar{o})$ be the identification map. Then the following is obtained in [1], §5.

Proposition 2. *Under the above assumptions the induced map* $p^{\Sigma}: SP(X,o)\to SP(X/A,\bar{o})$ *is a quasi-fibering*[1] *with a fiber* $(p^{\Sigma})^{-1}(\bar{o})$

1) A map $p: E\to B$ is said to be a quasi-fibering if p is onto and induces an isomorphism $p_*:\pi_n(E,F,x)\approx\pi_n(B,b)$ for any $b\in B$, $x\in F=p^{-1}(B)$ and $n\geqq 0$.

$=SP(A, o)$.

4. Axioms for homology. In this section we shall state the modified Eilenberg-Steenrod axioms in which the relative groups do not appear. These are different only in the dimension axiom from the axioms which are used by A. Dold and R. Thom in [1].

A homology theory H on a category $\mathfrak{A}$ is a collection of three functions as follows: The first function assigns to each space X (with base point) in $\mathfrak{A}$ and each integer q an abelian group $H_q(X)$. The second function assigns to each map $f: X\to Y$ (carrying the base point to the base point) in $\mathfrak{A}$ and each integer q a homomorphism $f_*: \mathrm{H}_q(X)\to H_q(Y)$. The third function assigns to each space X and each integer q an isomorphism $\varepsilon: H_q(X)\approx H_{q+1}(SX)$ which is called the suspension. Furthermore, the three functions are required to satisfy the following conditions:

1. *If f is the identity, then f_* is the identity.*
2. *If $f: X\to Y$ and $g: Y\to Z$, then $(gf)_*=g_*f_*$.*
3. *If two maps f, $g: X\to Y$ are homotopic, then $f_*=g_*$.*
4. *For each map $f: X\to Y$ and each integer q, the following diagram is commutative.*

$$\begin{array}{ccc} H_q(X) & \xrightarrow{\quad f_* \quad} & H_q(Y) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ H_{q+1}(SX) & \xrightarrow{\quad (Sf)_* \quad} & H_{q+1}(SY) \end{array}$$

5. *For each map $f: X\to Y$ and each integer q, the sequence*

$$H_q(X) \xrightarrow{f_*} H_q(Y) \xrightarrow{(Pf)_*} H_q(C_f)$$

is exact, where C_f is the mapping cone[2] *of f and $Pf: Y\to C_f$ is the* inclusion map.

6. $H_q(S^2)=0$, *for* $q\neq 2$.

Let X be a space and $\{X_\lambda\}$ be the system of compact subspaces. Let $i_\mu^\lambda: X_\lambda\to X_\mu$ and $i^\lambda: X_\lambda\to X$ be inclusions. Then $\{X_\lambda, i_\mu^\lambda\}$ is a directed system of spaces. Let $i_{\mu*}^\lambda: H_q(X_\lambda)\to H_q(X_\mu)$ be the induced isomorphism. Then $\{H_q(X_\lambda), i_{\mu*}^\lambda\}$ is a directed system of groups. Consider the limit group $\lim_\lambda H_q(X_\lambda)$. i^λ induces a homomorpism $i_*^\lambda: H_q(X_\lambda)\to H_q(X)$ and then a homomorphism $i_*: \lim_\lambda H_q(X_\lambda)\to H_q(X)$ is defined. Then we shall require the following condition.

7. *i_* is an isomorphism.*

Let $\mathfrak{C}$ denote the category of 1-connected CW-complexes (with base points). Then the following uniqueness theorem holds.

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2) The mapping cone C_f of a map $f: X\to Y$ is defined from the topological sum $(X\times I)\smile Y$ by identifying $(x, 1)$ with $f(x)$ and $(X\times 0)\smile(o\times I)$ with a point (the base point of C_f).

Theorem 3. *Let two homology theories $\{H_q(X), \varepsilon, f_*\}$ and $\{H'_q(X), \varepsilon', f_\#\}$ satisfying the axioms 1–7 be defined on the category $\mathfrak{C}$. Let a homorphism $j_2: H_2(S^2) \to H_2(S^2)$ be given. Then j_2 can be extended uniquely into the system of homomorphism $j = j(X, q): H_q(X) \to H'_q(X)$ so that for each map $f: X \to Y$ and the isomorphisms ε, ε' the commutativity holds in the following each diagram:*

$$\begin{array}{ccc} H_q(X) & \xrightarrow{f_*} & H_q(Y) \\ \downarrow j & & \downarrow j \\ H'_q(X) & \xrightarrow{f_\#} & H'_q(Y) \end{array} \qquad \begin{array}{ccc} H_q(X) & \xrightarrow{\varepsilon} & H_{q+1}(SX) \\ \downarrow j & & \downarrow j \\ H'_q(X) & \xrightarrow{\varepsilon'} & H'_{q+1}(SX) \end{array}$$

If j_2 is an isomorphism, then each j is an isomorphism.

The proof is parallel with Theorem 6.8 in [1], so we omit the proof.

5. The main theorem. It is known that the (reduced) singular homology theory satisfies the axioms 1–7 (described in §4) in the category $\mathfrak{C}$ of 1-connected CW-complexes (with base points) (cf. [2]). We shall prove the following

Theorem 4. *For each space X in $\mathfrak{C}$ and each integer q, define the abelian groups by*

$$H_q(X) = \begin{cases} 0, & \text{for } q \leqq 1, \\ \pi_2(SP(X, o)) \otimes G, & \text{for } q = 2, \\ \pi_q(SP(X, o); G), & \text{for } q \geqq 3. \end{cases}$$

For each map $f: X \to Y$ in $\mathfrak{C}$ and each integer q, define the homomorphisms

$$f_*: H_q(X) \to H_q(Y) = \begin{cases} 0, & \text{for } q \leqq 1, \\ f_*^\Sigma \otimes 1, & \text{for } q = 2, \\ f_*^\Sigma, & \text{for } q \geqq 3. \end{cases}$$

Then the axioms 1–7 are satisfied.

Proof. Since the addition defined in $SP(X)$ is continuous on a compact subset, $\pi_3(SP(X); G)$ is abelian (cf. [3], (3.13)) (throughout this proof we write $SP(X)$ instead of $SP(X, o)$). It is clear that the axioms 1–3 are satisfied.

Axiom 4. The identification map $p: CX \to CX/X = SX$ induces a quasi-fibering $p^\Sigma: SP(CX) \to SP(SX)$ with fiber $SP(X)$ according to Proposition 2. Then the exact sequence in Theorem 2 shows that $\partial': \pi_{q+1}(SP(SX); G) \to \pi_q(SP(X); G)$, $q \geqq 3$, is an isomorphism since CX is contractible. Let $\varepsilon = \partial'^{-1}$. The naturality of ε follows also from Theorem 2. Thus the proof is completed for $q \geqq 3$. We have $\pi_2(SP(SX)) * G \approx \pi_1(SP(X)) * G \approx H_1(X) * G = 0$ by Theorems 6.4 and 6.10 in [1]. Therefore by Proposition 1, $\varphi: \pi_3(SP(SX)) \otimes G \approx \pi_3(SP(SX); G)$. On the other hand, there exists an isomorphism $\varepsilon_0: \pi_2(SP(X)) \approx \pi_3(SP(SX))$ according to Theorem 6.4 in [1]. Define $\varepsilon = \varphi(\varepsilon_0 \otimes 1)$:

$\pi_2(SP(X))\otimes G\to\pi_3(SP(SX);G)$; then ε is an isomorphism, and is natural since ε_0 and φ are natural. When $q\leqq 1$ every group which appears in the axiom 4 is zero, and so we may take the unique homomorphism as ε.

Axiom 5. Let M_f be the mapping cylinder[3] of a map $f: X\to Y$. Since f and $Pf: X\to C_f$ are homotopically equivalent to the inclusion $f': X\to M_f$ and the identification map $p: M_f\to C_f$ respectively, the sequence

$$\pi_q(SP(X);G)\xrightarrow{f_*^\Sigma}\pi_q(SP(Y);G)\xrightarrow{(Pf)_*^\Sigma}\pi_q(SP(C_f);G)$$

is equivalent to the sequence

$$\pi_q(SP(X);G)\xrightarrow{f_*'^\Sigma}\pi_q(SP(M_f);G)\xrightarrow{p_*^\Sigma}\pi_q(SP(C_f);G),$$

for $q\geqq 3$. The latter is a part of the homotopy sequence of the quasi-fibering $p^\Sigma: SP(M_f)\to SP(C_f)$, and so it is exact. As for the case $q=2$, consider the ordinary homotopy sequence of p^Σ:

$$\cdots\to\pi_2(SP(X))\xrightarrow{f_*'^\Sigma}\pi_2(SP(M_f))\xrightarrow{p_*^\Sigma}\pi_2(SP(C_f))\xrightarrow{\partial'}\pi_1(SP(X))\to\cdots.$$

Since $\pi_1(SP(X))=0$, the sequence

$$\pi_2(SP(X))\otimes G\xrightarrow{f_*'^\Sigma\otimes 1}\pi_2(SP(M_f))\otimes G\xrightarrow{p_*^\Sigma\otimes 1}\pi_2(SP(C_f))\otimes G$$

is exact. Therefore we have the desired exact sequence in the similar way to the above case. When $q\leqq 1$ it is trivial.

Axiom 6. According to Theorem 6.10 in [1], $\pi_2(SP(S^2))\otimes G\approx H_2(S^2)\otimes G\approx Z\otimes G\approx G$. Since $\pi_q(SP(S^2))\otimes G=0=\pi_{q-1}(SP(S^2))*G$ for $q\geqq 3$ as above, Proposition 1 means $\pi_q(SP(S^2);G)=0$ for $q\geqq 3$. In case $q\leqq 1$, we have $H_q(S^2)=0$ by the assumption.

Axiom 7. Let $\{X_\lambda\}$ be a system of connected finite sub-complexes (with base point o) of X. Let $i_\mu^\lambda: X_\lambda\to X_\mu$ and $i^\lambda: X_\lambda\to X$ be inclusions. $\{i^{\lambda\Sigma}\}$ defines a 1–1 continuous map $i^\Sigma: \lim_\lambda SP(X_\lambda)\to SP(X)$. Since the inverse of i^Σ is continuous on any compact subset of $SP(X)$, we have an isomorphism $i_*^\Sigma: \lim_\lambda \pi_q(SP(X_\lambda);G)\approx\pi_q(SP(X);G)$. The proof of the theorem is thus completed.

As a concequence of Theorems 3 and 4 we have a generalization of Dold-Thom's isomorphism theorem.

Theorem 5. *If X is a 1-connected CW-complex with base point o, then there exist natural isomorphisms*

$$H_q(X;G)\approx\pi_q(SP(X,o);G),\ q\geqq 3,$$
$$H_2(X;G)\approx\pi_2(SP(X,o))\otimes G.$$

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3) The mapping cylinder M_f of a map $f: X\to Y$ is defined from the topological sum $(X\times I)\smile Y$ by identifying $(x,1)$ with $f(x)$ and $o\times I$ with a point (the base point of M_f).

References

[1] A. Dold und R. Thom: Quasifaserungen und unendliche symmetrische Produkte, Ann. of Math., **67**, 239–281 (1958).
[2] S. T. Hu: On axiomatic approach to homology theory without using the relative groups, Portugaliae Math., **19**, 211–225 (1960).