145. Homotopy Groups with Coefficients and a Generalization of Dold-Thom's Isomorphism Theorem. I

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1962)

1. **Introduction.** A. Dold and R. Thom established in [1] the existence of the following natural isomorphism

$$H_q(X)\approx\pi_q(SP(X,o)),\quad q\geqq 1,$$

for a connected CW-complex X with base point o, where $SP(X,o)$ denotes the infinite symmetric product of X. Professor K. Morita conjectured that there exists a natural isomorphism

$$H_q(X;G)\approx\pi_q(SP(X,o);G),\quad q\geqq 3,$$

for the homotopy groups with coefficients (in a finitely generated abelian group G) in the sense of Katuta [2]. In [3] we have proved that there exists the isomorphism above when X is a 1-connected countable simplicial complex. Here we shall show that the conjecture is true when X is a 1-connected CW-complex. The following theorem which was obtained in our previous paper [4] will play an important role in our proof.

Theorem 1. *Let spaces $E\supset F$, $B\supset C$ and a map $p:(E,F)\to(B,C)$ be given. If p is a weak homotopy equivalence of pairs of spaces, i.e. if p induces an isomorphism*

$$p_*:\pi_n(E,F)\approx\pi_n(B,C)\quad \textit{for any } n\geqq 0,$$

then for a CW-complex K the induced map $'p:(E^K,F^K)\to(B^K,C^K)$ is a weak homotopy equivalence of pairs of mapping spaces, i.e. $'p$ induces an isomorphism

$$'p_*:\pi_n(E^K,F^K)\approx\pi_n(B^K,C^K)\quad \textit{for any } n\geqq 0$$

where we mean a 1–1 correspondence by an isomorphism if $n\leqq 1$.

The author wishes here to express sincere thanks to Professor K. Morita for his helpful advices and suggestions.

2. **Homotopy groups with coefficients.** Throughout this paper we consider only spaces with base point and maps carrying the base point to the base point. Let G be a finitely generated abelian group. Y. Katuta defined homotopy groups with coefficients in G, $\pi_q(X;G)$, for $q\geqq 3$ and each space X as follows. Let us consider S^1 the unit circle in the complex number plane with 1 as the base point and let $\rho_m: S^1\to S^1$ be the map defined by $\rho_m(e^{i\theta})=e^{im\theta}$, for a positive integer m. Let $\rho_m^q: S^q\to S^q$ be the $(q-1)$-fold suspension $S^{q-1}\rho_m$ [1)] of ρ_m. Then

1) The suspension $Sf:SX\to SY$ of a map $f:X\to Y$ is defined by $Sf(s,x)=(s,f(x))$ for $s\in S^1$ and $x\in X$, and the q-fold suspension of f by $S^qf=S(S^{q-1}f)$ (see also the foot note[3)]).

ρ_m^q is a map of degree m. We define a q-dimensional CW-complex B_m^q, $q\geqq 2$, by attaching a q-cell $e^q=CS^{q-1}-S^{q-1}$ [2] to S^{q-1} by the map ρ_m^{q-1}. As is well known, for a finitely generated abelian group G we can write

$$G=\overbrace{Z+\cdots+Z}^{r\text{-fold}}+Z_{m_1}+\cdots+Z_{m_s},$$

where Z is an infinite cyclic group and Z_{m_i}, $1\leqq i\leqq s$, is a finite cyclic group of a prime power order m_i. Then q-dimensional CW-complex $P(G,q)$, $q\geqq 2$, is defined by

$$P(G,q)=\overbrace{S^q\vee\cdots\vee S^q}^{r\text{-fold}}\vee B_{m_1}^q\vee\cdots\vee B_{m_s}^q.$$

If G is a free abelian group of rank r, we define $P(G,0)$ as the discrete space consisting of $r+1$ points and $P(G,1)$ as $S^1\vee\cdots\vee S^1$ (r-fold).

Notice that $P(G,q)$, $q\geqq 3$, is a 1-connected space such that $H^q(P(G,q))\approx G$ and $H^i(P(G,q))=0$ for $i\neq q$. Clearly $P(G,q)=S^{q-2}P(G,2)(=S^qP(G,0)$, when G is free).[3] Then for a space X and $q\geqq 3$ ($q\geqq 1$ when G is free) we have $\Pi(P(G,q);x)=\Pi(S^{q-2}\# P(G,2);X)\approx \Pi(S^{q-2};X^{P(G,2)})=\pi_{q-2}(X^{P(G,2)})\ (=\pi_q(X^{P(G,0)})$ when G is free) (cf. [4], [5]), where $\Pi(K;X)$ denotes the set of homotopy classes of maps $K\to X$. We define $\pi_q(X;G)$, $q\geqq 3$, as $\Pi(P(G,q);X)$ which has a group structure by the above 1–1 correspondence and call it *the q-th homotopy group of X with coefficients in G*. Define $\pi_2(X;G)$ as a set $\Pi(P(G,2);X)$ ($\pi_0(X;G)$ as $\Pi(P(G,0);X)$ when G is free). Obviously, $\pi_q(X;Z)$ coincides with the ordinary q-th homotopy group $\pi_q(X)$. Hereafter we shall not consider the case G is free. From the definition $\pi_q(X;G)$ is abelian for $q\geqq 4$.

For a pair of spaces (X,A) and $q\geqq 3$ we define $\pi_q(X,A;G)=\Pi(CP(G,q-1),P(G,q-1);X,A)$. For $q\geqq 4$ it has a group structure and is called *the q-th relative homotopy group of* (X,A) *with coefficients in G*. It is abelian when $q\geqq 5$. It is easily seen that if A consists of the base point o of X, $\pi_q(X,o;G)$ may be identified with $\pi_q(X;G)$. Now we have the exact sequence

$$\cdots\to\pi_q(A;G)\xrightarrow{i_*}\pi_q(X;G)\xrightarrow{j_*}\pi_q(X,A;G)\xrightarrow{\partial}\pi_{q-1}(A;G)\to\cdots,$$

where i_* and j_* are induced by inclusions and ∂ by a restriction in the obvious way.

Theorem 2. *If* $p:(E,F)\to(B,b)$ *is a weak homotopy equivalence, then there exists the exact sequence*

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2) $CX=I\# X=I\times X/(0\times X)\smile(I\times o)$, the cone over X.

3) $SX=S^1\# X=S^1\times X/(s_0\times X)\smile(S^1\times o)$, the suspension of X. The q-fold suspension of X is defined by $S^qX=S(S^{q-1}X)$. Note that $S^qX=S^q\# X$, especially $S^q=S^{q-1}\# S^1=S^{q-1}S^1$.

$$\cdots \to \pi_q(F;G) \xrightarrow{i_*} \pi_q(E;G) \xrightarrow{p'_*} \pi_q(B;G) \xrightarrow{\partial'} \pi_{q-1}(F;G) \to \cdots .$$

Furthermore, the exact sequence is natural with respect to maps $f:(E,F)\to(E_1,F_1)$ *and* $g:B\to B_1$ *such that* $p_1f=gp$, *where* $p_1:(E_1,F_1)\to(B_1,b_1)$ *is another weak homotopy equivalence.*[4]

Proof. By Theorem 1 we have an isomorphism $'p_*:\pi_n(E^P,F^P)\approx\pi_n(B^P)$, for $n\geqq 0$ and $P=P(G,2)$. Now $\pi_n(E^P,F^P)=\Pi(CS^{n-1},S^{n-1};E^P,F^P)\overset{\theta}{\approx}\Pi(CS^{n-1}\# P(G,2),S^{n-1}\# P(G,2);E,F)=\Pi(CP(G,n+1),P(G,n+1);E,F)=\pi_{n+2}(E,F;G)$ and similarly $\pi_n(B^P)\overset{\theta}{\approx}\pi_{n+2}(B;G)$. Since these isomorphisms are natural (cf. [4], (2.3)), a map $p_*:\pi_q(E,F;G)\to\pi_q(B;G)$, $q\geqq 2$, defined by $p_*[f]=[pf]$ for $[f]\in\pi_q(E,F;G)$ satisfies $\theta\,'p_*=p_*\theta$ and hence it is an isomorphism. If we set $p_*j_*=p'_*$ and $\partial p_*^{-1}=\partial'$ in the exact sequence of the pair (E,F), we have the desired sequence.

The second part of the theorem is easily verified by the definitions of p'_* and ∂'.

The following proposition is proved in [2], Theorems 3.8 and 3.11 and in [3], (3.10).

Proposition 1. (*The universal coefficient theorem.*) *The following sequence is exact*:

$$0\to\pi_q(X)\otimes G\xrightarrow{\varphi}\pi_q(X;G)\xrightarrow{\psi}\pi_{q-1}(X)*G\to 0,^{5)}\quad q\geqq 3.$$

The exact sequence splits for $q\geqq 4$ *if* G *is* Z_p, *a finite cyclic group of an odd prime order* p, *and it is natural with respect to each map* $f:X\to Y$.

References

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[5] K. Morita: Note on mapping spaces, ibid., **32**, 671–675 (1956).

4) In [2] Y. Katuta proved the theorem if p is a fibering in the sense of Serre. But we need the theorem in case p is not necessarily a fibering.

5) $\otimes$ and $*$ denote the tensor and torsion products respectively. For the definitions of φ and ψ, see Theorem 3.8 in [2].