142. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. IV

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In this paper we shall discuss the curvilinear integrals of the functions $S(\lambda)$, $\Phi(\lambda)$, $\Psi(\lambda)$, which were defined in Theorem 1 established before, along a rectifiable closed Jordan curve containing an (uncountably or countably) infinite set of all the non-regular points of those functions inside itself [1].

Lemma A. Let $\{\lambda_\nu\}$ and $S(\lambda)$ be the same notations as those in Theorem 1 respectively; let ρ be any positive number satisfying the condition $\sup_\nu|\lambda_\nu|<\rho<\infty$; and let

$$a_n=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})\cos nt\,dt,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})\sin nt\,dt$$

for every positive integer n. Then each of $a_n^2+b_n^2$, $n=1,2,3,\cdots$, is constant independently of the value of ρ.

Proof. As already indicated by (9) in my preceding paper,

$$S\left(\frac{\rho e^{it}}{\kappa}\right)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)\left(\frac{e^{it}}{\kappa}\right)^n+\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{it}}\right)^n,$$

where $\sup_\nu|\lambda_\nu|<\rho<\infty$, $0<\kappa<1$, and $a_0=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})dt$; and moreover the two series on the right-hand side both converge absolutely and uniformly for any κ with $0<\kappa<1$. If we now put

$$a_n'=\frac{1}{\pi}\int_0^{2\pi}S\left(\frac{\rho e^{it}}{\kappa}\right)\cos nt\,dt,$$

$$b_n'=\frac{1}{\pi}\int_0^{2\pi}S\left(\frac{\rho e^{it}}{\kappa}\right)\sin nt\,dt,$$

then we can find with the help of the above equality that

$$a_n'+ib_n'=\frac{1}{\pi}\int_0^{2\pi}S\left(\frac{\rho e^{it}}{\kappa}\right)e^{int}\,dt$$
$$=\kappa^n(a_n+ib_n)$$

and

$$a_n'-ib_n'=\frac{1}{\pi}\int_0^{2\pi}S\left(\frac{\rho e^{it}}{\kappa}\right)e^{-int}\,dt$$
$$=\kappa^{-n}(a_n-ib_n).$$

Consequently we obtain the relations $a_n'^2+b_n'^2=a_n^2+b_n^2$, $n=1,2,3,\cdots$, for every ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$ and every κ with $0<\kappa<1$.

Thus the present lemma has been proved.

Theorem 10. Let $\{\lambda_\nu\}$, $S(\lambda)$, $R(\lambda)$, and Γ be the same notations as those in Theorem 1 respectively, and let K_1 denote the constant value $a_1^2+b_1^2$ shown in Lemma A. If $R'(0)\neq 0$, then

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda=\frac{K_1}{4R'(0)}.$$

Proof. Let ρ be a positive number such that the circle C with center at the origin and radius ρ contains Γ inside itself. Since $S(\lambda)$ is regular on the closed domain surrounded by the two simple curves C and Γ in accordance with the hypothesis concerning $S(\lambda)$, it is seen by Cauchy's integral theorem that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda&=\frac{1}{2\pi i}\int_C S(\lambda)d\lambda\\&=\frac{\rho}{2\pi}\int_0^{2\pi}S(\rho e^{it})e^{it}\,dt\\&=\frac{\rho}{2}(a_1+ib_1),\end{aligned}$$

where the curvilinear integrations are taken in the counterclockwise direction. On the other hand,

$$\begin{aligned}a_1-ib_1&=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})e^{-it}\,dt\\&=\frac{\rho}{\pi i}\int_C\frac{S(\lambda)}{\lambda^2}d\lambda\\&=2\rho R'(0)\end{aligned}$$

by virtue of the fact that

$$\frac{1}{2\pi i}\int_C\frac{S(\lambda)}{\lambda^{k+1}}d\lambda=\frac{R^{(k)}(0)}{k!}\qquad(k=0,1,2,\cdots),$$

as already shown [2]. These results enable us to conclude that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda&=\frac{\rho(a_1^2+b_1^2)}{2(a_1-ib_1)}\\&=\frac{K_1}{4R'(0)},\end{aligned}$$

as we were to prove.

Remark. When $R'(0)=0$, consider the function $T(\lambda)=S(\lambda)+\lambda$ by way of example. By following the argument used in the proof of Theorem 10, it can then be found that

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda=\frac{1}{2\pi i}\int_C T(\lambda)d\lambda$$

$$=\frac{K_1'}{4},$$

where K_1' denotes the constant $\left\{\frac{1}{\pi}\int_0^{2\pi}T(\rho e^{it})\cos t\,dt\right\}^2+\left\{\frac{1}{\pi}\int_0^{2\pi}T(\rho e^{it})\sin t\,dt\right\}^2$ independent of the value of ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$.

Theorem 11. Let $\{\lambda_\nu\}$, $S(\lambda)$, $\{c_\alpha^{(\nu)}\}$, and Γ be the same notations as those in Theorem 1 respectively. If all the accumulation points of $\{\lambda_\nu\}$ form a countably infinite set, then

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda=\sum_\nu c_1^{(\nu)}.$$

Proof. Since, by hypothesis, the set of all the accumulation points of $\{\lambda_\nu\}$ is countably infinite, the second principal part of $S(\lambda)$ vanishes as we have already pointed out in the earlier discussion. By reference to Cauchy's integral theorem we have therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda&=\frac{1}{2\pi i}\int_\Gamma\{\Phi(\lambda)+R(\lambda)\}d\lambda\\&=\frac{1}{2\pi i}\int_\Gamma\Phi(\lambda)d\lambda\\&=\frac{1}{2\pi i}\int_\Gamma\sum_{\alpha=1}^{m}\sum_\nu\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha},\end{aligned}$$

denoting by $\Phi(\lambda)$ the first principal part of $S(\lambda)$. On the other hand, as we indicated in the course of the proof of Theorem 1, $\Phi(\lambda)$ is expressible in the form $\Phi(\lambda)=\sum_{\alpha=1}^{m}((\lambda I-N)^{-\alpha}f_\alpha,\bar{f}_\alpha)$, where N is an appropriately chosen bounded normal operator with point spectrum $\{\lambda_\nu\}$ such that its continuous spectrum consists of accumulation points of $\{\lambda_\nu\}$ alone and the elements $f_\alpha,\bar{f}_\alpha$ are appropriately chosen in the subspace determined by all the eigenelements of N [2]. Since, moreover, each of $(\lambda I-N)^{-\alpha}$, $\alpha=1,2,\cdots,m$, is differentiable at any point λ belonging to the resolvent set of N, $\Phi(\lambda)$ is regular on the entire complex λ-plane except for the set $\{\lambda_\nu\}$ and its accumulation points. Using again the circle C, defined before and oriented positively, and remembering that $\sum_\nu|c_\alpha^{(\nu)}|<\infty$, $\alpha=1,2,\cdots,m$, by hypothesis and $\sup_\nu|\lambda_\nu|<|\lambda|$ for every $\lambda\in C$, we obtain therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)d\lambda&=\frac{1}{2\pi i}\int_C\sum_{\alpha=1}^{m}\sum_\nu\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha}d\lambda\quad(C:|\lambda|=\rho)\\&=\sum_\nu\frac{1}{2\pi i}\int_C\sum_{\alpha=1}^{m}\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha}d\lambda\\&=\sum_\nu c_1^{(\nu)}\end{aligned}$$

by means of the fact that term by term integration of series is allowed on account of its uniform convergence.

With this result, the proof of the theorem is complete.

Remark. In the case that all the accumulation points of $\{\lambda_\nu\}$ form an uncountably infinite set, the second principal part $\Psi(\lambda)$ of $S(\lambda)$ never vanishes and moreover it follows immediately from the results established in Theorems 10 and 11 that the relation

$$\frac{1}{2\pi i}\int_\Gamma \Psi(\lambda)\,d\lambda=\frac{K_1}{4R'(0)}-\sum_\nu c_1^{(\nu)}$$

holds under the condition $R'(0)\neq 0$.

Theorem 12. Let $\{\lambda_\nu\}$, $S(\lambda)$, $R(\lambda)$, and Γ be the same notations as those in Theorem 1 respectively, and let K_n denote the constant value $a_n^2+b_n^2$ shown in Lemma A. If $P(\lambda)$ is a polynomial in λ of precisely the degree M and if $R^{(n)}(0)\neq 0$ for $n=1, 2, 3,\cdots, M+1$, then

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)P(\lambda)d\lambda=\frac{1}{4}\sum_{\mu=0}^{M}\frac{(\mu+1)P^{(\mu)}(0)K_{\mu+1}}{R^{(\mu+1)}(0)},$$

where $P^{(0)}(0)$ denotes $P(0)$.

Proof. Let C denote the same circle as that defined at the beginning of the proof of Theorem 10. Then the reasoning used in the proof of Theorem 10 can be applied without change to show that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)\lambda^\mu d\lambda&=\frac{1}{2\pi i}\int_C S(\lambda)\lambda^\mu d\lambda \quad (\mu=0, 1, 2,\cdots, M)\\&=\frac{\rho^{\mu+1}}{2\pi}\int_0^{2\pi} S(\rho e^{it})e^{i(\mu+1)t}dt\\&=\frac{1}{2}\rho^{\mu+1}(a_{\mu+1}+ib_{\mu+1})\end{aligned}$$

and

$$\begin{aligned}a_{\mu+1}-ib_{\mu+1}&=\frac{1}{\pi}\int_0^{2\pi} S(\rho e^{it})e^{-i(\mu+1)t}dt\\&=\frac{\rho^{\mu+1}}{\pi i}\int_C \frac{S(\lambda)}{\lambda^{\mu+2}}d\lambda\\&=\frac{2\rho^{\mu+1}R^{(\mu+1)}(0)}{(\mu+1)!}.\end{aligned}$$

Since, by hypothesis, $R^{(n)}(0)\neq 0$ for $n=1, 2, 3,\cdots, M+1$, the just established results permit us to assert that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma S(\lambda)\lambda^\mu d\lambda&=\frac{\rho^{\mu+1}K_{\mu+1}}{2(a_{\mu+1}-ib_{\mu+1})}\\&=\frac{(\mu+1)!\,K_{\mu+1}}{4R^{(\mu+1)}(0)}.\end{aligned}$$

We thus obtain

$$\frac{1}{2\pi i}\int_\Gamma S(\lambda)P(\lambda)d\lambda=\sum_{\mu=0}^{M}\frac{1}{2\pi i}\int_\Gamma \frac{P^{(\mu)}(0)S(\lambda)\lambda^\mu}{\mu!}d\lambda$$

$$=\frac{1}{4}\sum_{\mu=0}^{M}\frac{(\mu+1)P^{(\mu)}(0)K_{\mu+1}}{R^{(\mu+1)}(0)},$$

as we wished to prove.

References

[1] S. Inoue: Some applications of the functional-representations of normal operators in Hilbert spaces, Proc. Japan Acad., **38**, 263–268 (1962).

[2] ——: Ibid., 266.