141. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. III*

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In this paper we shall turn to the problem of finding the extended Fourier-series expansion corresponding to each of the functions $S(\lambda)$, $\Phi(\lambda)$, $\Psi(\lambda)$, and $R(\lambda)$ defined in the statement of Theorem 1 [cf. Vol. 38, No. 6 (1962), pp. 263–268].

Theorem 6. Let $\{\lambda_\nu\}$, $S(\lambda)$, and $R(\lambda)$ be the same notations as those in Theorem 1 respectively. Then, for every ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$ and every κ with $0\leqq\kappa<\infty$,

$$(7)\qquad R(\kappa\rho e^{i\theta})=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa e^{i\theta})^n \quad (\theta\text{: variable}),$$

where

$$(8)\qquad \begin{cases} a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}S(\rho e^{it})\cos nt\,dt \\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}S(\rho e^{it})\sin nt\,dt \end{cases} \quad (n=0,1,2,3,\cdots)$$

and the series on the right-hand side converges absolutely and uniformly.

Proof. It follows from Theorem 1 that

$$\begin{aligned}\frac{1}{2}(a_n-ib_n)&=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})e^{-int}dt \quad (n=0,1,2,3,\cdots)\\ &=\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{S(\lambda)\rho^n}{\lambda^{n+1}}d\lambda\\ &=\frac{R^{(n)}(0)\rho^n}{n!},\end{aligned}$$

where $0!$ and $R^{(0)}(0)$ denote 1 and $R(0)$ respectively, so that

$$\begin{aligned}\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa e^{i\theta})^n&=\sum_{n=0}^{\infty}\frac{R^{(n)}(0)}{n!}(\kappa\rho e^{i\theta})^n \quad (0\leqq\kappa<\infty)\\ &=R(\kappa\rho e^{i\theta}).\end{aligned}$$

In addition, the absolute and uniform convergence of the series on the right-hand side of (7) is a direct consequence of the hypothesis that $R(\lambda)$ is regular on the domain $\{\lambda:|\lambda|<\infty\}$.

Theorem 7. Let $\{\lambda_\nu\}$, $S(\lambda)$, and $R(\lambda)$ be the same notations as before. Then, for every ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$ and every κ with $0<\kappa<1$,

$$(9)\qquad S\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)\left(\frac{e^{i\theta}}{\kappa}\right)^n+\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{i\theta}}\right)^n,$$

where a_n and b_n are given by (8) and the two series on the right-hand side both converge absolutely and uniformly.

Proof. As already demonstrated in my preceding paper, the equality

$$(10)\quad S\left(\frac{\rho e^{i\theta}}{\kappa}\right)-R\left(\frac{\rho e^{i\theta}}{\kappa}\right)+R(\kappa\rho e^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$

holds for every ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$ and every κ with $0<\kappa<1$. Moreover, in the same manner as that for the real Poisson integral, we can find that the complex Poisson integral on the right-hand side of (10) is expansible in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\kappa^n(a_n \operatorname{con} n\theta+b_n\sin n\theta)$$

where a_n and b_n are given by (8). By applying this result and Theorem 6 to (10) we have

$$S\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\kappa^n(a_n\cos n\theta+b_n\sin n\theta)+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)\left(\frac{e^{i\theta}}{\kappa}\right)^n$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa e^{i\theta})^n\quad(0<\kappa<1),$$

where the three series on the right-hand side converge absolutely and uniformly on account of the fact that the sets $\{a_n\}$ and $\{b_n\}$ both are bounded and that $\frac{1}{2}(a_n-ib_n)=R^{(n)}(0)\rho^n/n!$ for $n=0,1,2,3,\cdots$; and by direct calculation it is easily found that the just established expansion of $S\left(\frac{\rho e^{i\theta}}{\kappa}\right)$ is rewritten in the form of the right-hand side of (9). Moreover it is clear that the last series on the right of (9) converges absolutely and uniformly for any κ with $0<\kappa<1$.

With these results the proof of the theorem is complete.

Theorem 8. Let $\{\lambda_\nu\}$ and $S(\lambda)$ be the same notations as before. If all the accumulation points of $\{\lambda_\nu\}$ form a countable set, then the first principal part $\Phi(\lambda)$ of $S(\lambda)$ is expansible in the form

$$\Phi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{i\theta}}\right)^n\quad(0<\kappa<1,\ \sup_\nu|\lambda_\nu|<\rho<\infty),$$

where a_n and b_n are given by (8) and the series on the right-hand side converges absolutely and uniformly.

Proof. By the hypothesis on the set $\{\lambda_\nu\}$ we have

$$\Phi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\sum_{\alpha=1}^{m}\sum_{\nu}c_\alpha^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_\nu\right)^{-\alpha}$$
$$=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt-R(\kappa\rho e^{i\theta})$$

$$(0<\kappa<1, \sup_{\nu}|\lambda_\nu|<\rho<\infty),$$

as already shown in (2) of my preceding paper. Consequently it is found immediately from the course of the proof of Theorem 7 that

$$\Phi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\kappa^n(a_n\cos n\theta+b_n\sin n\theta)-\frac{a_0}{2}-\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa e^{i\theta})^n$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{i\theta}}\right)^n,$$

where the series on the right of the final relation converges absolutely and uniformly.

Remark. If all the accumulation points of $\{\lambda_\nu\}$ form an uncountable set, the second principal part $\Psi(\lambda)$ is expansible in the form

$$\Psi\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{i\theta}}\right)^n-\sum_{\alpha=1}^{m}\sum_{\nu}c_\alpha^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_\nu\right)^{-\alpha}$$
$$(0<\kappa<1, \sup_{\nu}|\lambda_\nu|<\rho<\infty),$$

as will be seen immediately from (1) in the preceding paper.

Theorem 9. Let $\{\lambda_\nu\}$ and $S(\lambda)$ be the same notations as those in Theorem 1 respectively. If there are a positive number σ with $\sup_{\nu}|\lambda_\nu|<\sigma<\infty$ and a countably infinite set of points $r_j e^{i\theta_j}$ with $\sup_j r_j<\sigma$ such that

$$\int_0^{2\pi}\frac{S(\sigma e^{it})}{\sigma e^{it}-r_j e^{i\theta_j}}dt=0 \quad (j=1, 2, 3, \cdots),$$

then the relations

$$\frac{1}{\pi}\int_0^{2\pi}\Re[S(\rho e^{it})]\cos nt\,dt=-\frac{1}{\pi}\int_0^{2\pi}\Im[S(\rho e^{it})]\sin nt\,dt,$$
$$\frac{1}{\pi}\int_0^{2\pi}\Re[S(\rho e^{it})]\sin nt\,dt=\frac{1}{\pi}\int_0^{2\pi}\Im[S(\rho e^{it})]\cos nt\,dt$$

hold for every positive integer n and every ρ with $\sup_{\nu}|\lambda_\nu|<\rho<\infty$, and $S(\lambda)$ is expansible in the form

$$S\left(\frac{\rho e^{i\theta}}{\kappa}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\left(\frac{\kappa}{e^{i\theta}}\right)^n \quad (0<\kappa<1),$$

where a_n, $n=0, 1, 2, \cdots$, are given by (8).

Proof. As already proved at the beginning of the proof of Corollary 1 in my preceding paper, it is found by hypothesis that the ordinary part $R(\lambda)$ of $S(\lambda)$ is a constant which will be denoted by C and hence that

$$S\left(\frac{\rho e^{it}}{\kappa}\right)=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{i\varphi})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(t-\varphi)}d\varphi \quad (0<\kappa<1, \sup_{\nu}|\lambda_\nu|<\rho<\infty)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\kappa^n(a_n\cos nt+b_n\sin nt), \tag{11}$$

where a_n and b_n are given by (8). Moreover, on the one hand,

$$\begin{aligned}C&=R(0)\\&=\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{S(\lambda)}{\lambda}d\lambda\\&=\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})dt\\&=\frac{a_0}{2},\end{aligned}$$

and on the other hand,

$$\begin{aligned}C&=R(z)\quad\left(z=re^{i\theta},\ r<\rho,\ \kappa=\frac{r}{\rho}\right)\\&=\frac{1}{2\pi i}\int_{|\lambda|=\frac{\rho}{\kappa}}\frac{S(\lambda)}{\lambda-z}d\lambda\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{S\left(\frac{\rho e^{it}}{\kappa}\right)\frac{\rho e^{it}}{\kappa}}{\frac{\rho e^{it}}{\kappa}-re^{i\theta}}dt\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{S\left(\frac{\rho e^{it}}{\kappa}\right)}{1-\kappa^2e^{i(\theta-t)}}dt.\end{aligned}$$

By applying (11) and the just indicated relation $\frac{a_0}{2}=C$ to the final relation, we obtain

$$\begin{aligned}C&=\frac{1}{2\pi}\int_0^{2\pi}\{C+\sum_{n=1}^{\infty}\kappa^n(a_n\cos nt+b_n\sin nt)\}\times\\&\qquad\{1+\sum_{n=1}^{\infty}(\kappa^2e^{i\theta})^n(\cos nt-i\sin nt)\}dt\\&=C+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa^3e^{i\theta})^n\end{aligned}$$

and hence

$$\sum_{n=1}^{\infty}(a_n-ib_n)(\kappa^3e^{i\theta})^n\equiv 0.$$

If, for simplicity, we now make use of abbreviations

$$\begin{aligned}A_n&=\frac{1}{\pi}\int_0^{2\pi}\Re[S(\rho e^{it})]\cos nt\,dt,\\B_n&=\frac{1}{\pi}\int_0^{2\pi}\Im[S(\rho e^{it})]\cos nt\,dt,\\C_n&=\frac{1}{\pi}\int_0^{2\pi}\Re[S(\rho e^{it})]\sin nt\,dt,\\D_n&=\frac{1}{\pi}\int_0^{2\pi}\Im[S(\rho e^{it})]\sin nt\,dt,\end{aligned}$$

the just established identity is rewritten, as follows:

$$\sum_{n=1}^{\infty}\kappa^{3n}\{(A_n+D_n)+i(B_n-C_n)\}(\cos n\theta+i\sin n\theta)\equiv 0.$$

Accordingly the two identities

$$\sum_{n=1}^{\infty}\kappa^{3n}\{(A_n+D_n)\cos n\theta-(B_n-C_n)\sin n\theta\}\equiv 0,$$

$$\sum_{n=1}^{\infty}\kappa^{3n}\{(A_n+D_n)\sin n\theta+(B_n-C_n)\cos n\theta\}\equiv 0$$

hold for every κ with $0<\kappa<1$, so that

$$(A_n+D_n)\cos n\theta\equiv(B_n-C_n)\sin n\theta,$$
$$(A_n+D_n)\sin n\theta\equiv-(B_n-C_n)\cos n\theta$$

for $n=1, 2, 3,\cdots$. From the final two systems of identities, it follows at once that $A_n=-D_n$ and $B_n=C_n$ for $n=1, 2, 3,\cdots$, and hence that $a_n=ib_n$ for $n=1, 2, 3,\cdots$.

Furthermore we can easily find that an application of the system of relations $a_n=ib_n$, $n=1, 2, 3,\cdots$, to (11) yields the desired expansion of $S\left(\frac{\rho e^{i\theta}}{\kappa}\right)$.

The proof of the theorem has thus been finished.

Remark. It can be verified without difficulty that, if there are a positive number σ with $\sup_{\nu}|\lambda_\nu|<\sigma<\infty$ and a countably infinite set of points z_j with $\sup_{j}|z_j|<\sigma$ such that the integrals

$$\int_{|\lambda|=\sigma}\frac{S(\lambda)}{(\lambda-z_j)^{\mu+1}}d\lambda \quad (j=1, 2, 3,\cdots)$$

assume the same value, not zero, then results analogous to those of Theorem 9 are established for the μ-th derivative $S^{(\mu)}(\lambda)$ on the domain $\{\lambda: \sup_{\nu}|\lambda_\nu|<|\lambda|<\infty\}$. The same is true of the case where there exists a positive number σ with $\sup_{\nu}|\lambda_\nu|<\sigma<\infty$ such that

$$\int_{|\lambda|=\sigma}\frac{S(\lambda)}{\lambda^{\mu+1}}d\lambda\neq 0, \quad \int_{|\lambda|=\sigma}\frac{S(\lambda)}{\lambda^{\mu+p}}d\lambda=0 \quad (p=2, 3, 4,\cdots).$$

In either case it turns out, in fact, that $R(\lambda)$ is a polynomial in λ of precisely the degree μ, the ordinary part of $S^{(\mu)}(\lambda)$ is given by $R^{(\mu)}(\lambda)$, and the set of non-regular points of $S^{(\mu)}(\lambda)$ consists of the set $\{\lambda_\nu\}$ and its accumulation points.