140. Remark on Lehto's Paper "A Generalization of Picard's Theorem"

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1. Let E denote a sequence $\{a_n\}$ of points tending to the point at infinity and let $\overline{E}$ be the union of E and the point at infinity. If a function $f(z)$ is single-valued and meromorphic outside $\overline{E}$ and possesses at least one essential singularity in E, then of course $f(z)$ takes every value infinitely often outside $\overline{E}$ except for at most two. But if $f(z)$ possesses just one essential singularity at infinity, then it may omit more than two values. For instance, if we take as E the union of 0-points and 1-points of a non-rational entire function $f(z)$ for $z \neq \infty$, $f(z)$ omits outside $\overline{E}$ three values 0, 1 and ∞.

Lehto's main results [4] are as follows:

Theorem A. *Let $f(z)$ be single-valued and meromorphic outside $\overline{E}$ and possess at least one essential singularity in $\overline{E}$. If the points of E satisfy the condition*

$$(*) \qquad (\log|a_\nu|)^{2+\delta} = O(\log|a_{\nu+1}|) \qquad (\delta>0),$$

then $f(z)$ takes every value infinitely often outside $\overline{E}$ with possible two exceptions.

Theorem B. *Let $f(z)$ be a non-rational entire function for $z \neq \infty$. If the points of E satisfy the condition*

$$(**) \qquad |a_\nu/a_{\nu+1}| = O(\nu^{-2}),$$

then $f(z)$ takes every finite value infinitely often outside $\overline{E}$ except for at most one.

In this note, we shall show that the condition (**) in Theorem B can be relaxed considerably and give a remark to Picard's theorem on ends.

2. First we shall give a lemma which is a consequence of Schottky-Bohr-Landau's theorem.

Lemma. *Let $f(z)$ be a regular function in a ring domain A: $a<|z|<b$ such that*

$$f(z) \neq 0, 1 \quad \textit{and} \quad \min_{|z|=\sqrt{ab}} |f(z)| \leqq k$$

for a positive number k. Then we have

$$\max_{|z|=\sqrt{ab}} \log\log|f(z)| \leqq (\log K)\left(\frac{1}{\log b/a} + \frac{1}{4\pi}\right) + \log\log(k+2),$$

where K is a positive constant.

Proof. Consider the function $g(\zeta)=f(e^{\zeta})$ $(\zeta=\xi+i\eta)$. Then $g(\zeta)$ is a regular function in the strip domain $\log a<\xi<\log b$ and has the properties that

$$g(\zeta)\neq 0,1 \quad\text{and}\quad g(\zeta+2n\pi i)=g(\zeta) \qquad (n:\text{ integer}).$$

From this second property of g, we can find two points ζ_1 and ζ_2 on the line $L:\xi=\frac{1}{2}\log ab$ with the conditions that

$$|g(\zeta_1)|=\min_{\zeta\in L}|g(\zeta)|,\quad |g(\zeta_2)|=\max_{\zeta\in L}|g(\zeta)|$$

and

$$|\zeta_1-\zeta_2|\leqq\pi.$$

Here we note that

$$|g(\zeta_1)|=\min_{|z|=\sqrt{ab}}|f(z)| \quad\text{and}\quad |g(\zeta_2)|=\max_{|z|=\sqrt{ab}}|f(z)|.$$

We use the following Bohr-Landau's theorem [1] which is a precise form of Schottky's theorem.

Let $f(z)$ be a regular function in a disc $|z|<R$ such that

$$f(z)\neq 0,1 \quad \textit{and} \quad |f(0)|\leqq k.$$

Then for every z satisfying that $|z|\leqq\theta R$ $(0<\theta<1)$,

$$\log|f(z)|\leqq\frac{D\log(k+2)}{1-\theta},$$

where D is a positive constant.

Set $B=\max(1,D)$. Using the above theorem for $\theta=1/2$ n-times, we can see easily that

$$\log|g(\zeta)|\leqq 2B(2B+1)^{n-1}\log(k+2)$$

for every ζ on L with the property that $|\zeta-\zeta_1|\leqq\frac{n}{4}\log b/a$. Since $|\zeta_2-\zeta_1|\leqq\pi\leqq\left(\left[\frac{4\pi}{\log b/a}\right]+1\right)\frac{1}{4}\log b/a$,

$$\log|g(\zeta_2)|\leqq 2B(2B+1)^{\frac{4\pi}{\log b/a}}\log(k+2)$$
$$\leqq(2B+1)^{\frac{4\pi}{\log b/a}+1+\frac{\log\log(k+2)}{\log(2B+1)}},$$

where we denote by $[t]$ the greatest integer not exceeding t. Setting $K=(2B+1)^{4\pi}$, we have

$$\max_{|z|=\sqrt{ab}}\log\log|f(z)|=\log\log|g(\zeta_2)|$$
$$\leqq(\log K)\left(\frac{1}{\log b/a}+\frac{1}{4\pi}+\frac{\log\log(k+2)}{\log K}\right).$$

Our proof is complete.

3. Now we prove

Theorem 1. *Let $f(z)$ be a non-rational entire function for $z\neq\infty$. Then $f(z)$ takes every finite value outside $\overline{E}$ infinitely often except for at most one, if the points of E satisfy the condition*

$$\log|a_{\nu+1}/a_\nu|\geqq m(\nu),$$

where $m(\nu)$ $(\nu=1,2,\cdots)$ are positive numbers such that

$$\overline{\lim_{n\to\infty}} \frac{K^{\frac{1}{m(n)}}}{\sum_{\nu=1}^{n} m(\nu)} < +\infty.$$

Proof. Contrary to our assertion, we shall suppose that $f(z)$ takes only finitely often more than one finite values outside $\overline{E}$. Then we may assume without loss of generality that $f(z)$ omits values 0 and 1 in $\{z; |z|>R\} \frown CE$[1] for sufficiently large R. Since any non-rational entire function has at most two exceptional values, we may assume also that E contains an infinite number of 1-points of $f(z)$. Let $\{a_{\nu_i}\}$ $(i=1,2,\cdots)$ be those satisfying $|a_{\nu_i}|>R$.

Consider the domain $D_i: r_i=\sqrt{|a_{\nu_{i-1}}a_{\nu_i}|}<|z|<r_i'=\sqrt{|a_{\nu_i}a_{\nu_{i+1}}|}$ for each i. Then $f(z)\neq 0$ in D_i and hence we have

$$\min_{|z|=r_i}|f(z)|\leqq 1 \quad \text{or} \quad \min_{|z|=r_i'}|f(z)|\leqq 1$$

by the minimum principle. Therefore we see by our lemma that

$$\max_{|z|=r_i}\log|f(z)|\leqq K^{\frac{1}{\log|a_{\nu_i}/a_{\nu_i-1}|}+\frac{1}{4\pi}+\frac{\log 3}{\log K}}\leqq(\log 3)K^{\frac{1}{4\pi}+\frac{1}{m(\nu_i-1)}}$$

or

$$\max_{|z|=r_i'}\log|f(z)|\leqq(\log 3)K^{\frac{1}{4\pi}+\frac{1}{m(\nu_i)}}.$$

On the other hand

$$\log r_i \geqq \sum_{\nu=1}^{\nu_i-2} m(\nu)+\log|a_1| \quad \text{and} \quad \log r_i' \geqq \sum_{\nu=1}^{\nu_i-1} m(\nu)+\log|a_1|,$$

and we have

$$(\log 3)K^{\frac{1}{4\pi}}\frac{K^{\frac{1}{m(\nu_i-1)}}}{\sum_{\nu=1}^{\nu_i-2} m(\nu)+\log|a_1|}\geqq\frac{\max_{|z|=r_i}\log|f(z)|}{\log r_i}$$

or

$$(\log 3)K^{\frac{1}{4\pi}}\frac{K^{\frac{1}{m(\nu_i)}}}{\sum_{\nu=1}^{\nu_i-1} m(\nu)+\log|a_1|}\geqq\frac{\max_{|z|=r_i'}\log|f(z)|}{\log r_i'}$$

Hence we can conclude from our assumption that

$$+\infty>\underline{\lim_{r\to\infty}}\frac{\max_{|z|=r}\log|f(z)|}{\log r}\geqq\underline{\lim_{r\to\infty}}\frac{T(r)}{\log r},$$

where $T(r)$ is the characteristic function of $f(z)$. By a well-known theorem on meromorphic functions,[2] $f(z)$ must be a rational function; this is a contradiction. Thus our theorem is established.

Corollary. *Let $f(z)$ be a non-rational entire function for $z\neq\infty$. Then $f(z)$ takes every finite value infinitely often outside $\overline{E}$ with possible one exception, if the points of E satisfy the condition*

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1) We denote by CA the complement of a set A with respect to the complex plane.

2) See R. Nevanlinna [5], p. 174.

$$(**)' \qquad \log|a_{\nu+1}/a_\nu| \geqq (\log\nu)^{\delta-1} \quad (\delta>0,\ \nu\geqq 2).$$

For we can see by an easy computation that

$$\lim_{n\to\infty} \frac{K^{(\log n)^{1-\delta}}}{\sum_{\nu=2}^{n}(\log\nu)^{\delta-1}}=0.$$

The condition $(**)'$ is considerably weaker than the condition $(**)$.

4. Last we shall give a remark to Picard's theorem on ends in the sense of Heins [2]. Let R be a Riemann surface which belongs to the class O_G and has precisely one ideal boundary component in the sense that the complement of every compact subset has precisely one component which is not relatively compact. Heins called any subregion of R with a compact complement an "end" of R and showed that, for any single-valued meromorphic function on an end, the cluster set at the ideal boundary is the whole w-plane or reduces to one point. Further he showed that Picard's theorem holds on each end of a Riemann surface R if R admits a sequence $\{A_\nu\}$ of ring domains on R such that $A_{\nu+1}$ separates A_ν from the ideal boundary and $\mathrm{mod}\, A_\nu \geqq a>0$ for each ν, where a is a positive constant. Here we remark that the last condition can be relaxed, that is, the following holds.

Theorem 2. *Let R be a Riemann surface admitting a sequence $\{A_\nu\}$ of ring domains such that, for each ν, $A_{\nu+1}$ separates A_ν from the ideal boundary and*

$$\mathrm{mod}\, A_\nu \geqq (\log\nu)^{\delta-1} \qquad (\delta>0,\ \nu\geqq 2).$$

Then every single-valued meromorphic function $f(p)$ on any end of R takes all values infinitely often with the exception of at most two, if the cluster set at the ideal boundary is the whole w-plane.

Proof. Contrary, suppose that $f(z)$ omits three values 0, 1 and ∞ on an end Ω. We note that these three values are asymptotic values along curves tending to the ideal boundary and hence

$$\min_{p\in\Gamma_\nu}|f(p)|\leqq 1 \qquad \text{for sufficiently large } \nu,$$

where Γ_ν is the closed curve dividing A_ν into two ring domains with the same harmonic modulus $\mathrm{mod}\, A/2$. Therefore by Lemma we have

$$M_\nu = \max_{p\in\Gamma_\nu}\log|f(p)| \leqq (\log 3)K^{\frac{1}{4\pi}+(\log\nu)^{1-\delta}}$$

and hence

$$0=\lim \frac{(\log 3)^2 K^{\frac{1}{2\pi}}K^{2(\log n)^{1-\delta}}}{\sum_{\nu=2}^{n-1}(\log\nu)^{\delta-1}} \geqq \varlimsup_{n\to\infty}\frac{(M_n)^2}{\sum_{\nu=2}^{n-2}\mathrm{mod}\, A_\nu}.$$

By a theorem of the Phragmén-Lindelöf type (Kuroda [3], Theorem 2), $f(z)$ must be bounded; this contradiction proves the theorem.

References

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[4] O. Lehto: A generalization of Picard's theorem, Arkiv Mat., **3**(45), 495–500 (1958).
[5] R. Nevanlinna: Eindeutige analytische Funktionen, Springer, Berlin (1936).