# 139. Unique Continuation Theorem of Elliptic Systems of Partial Differential Equations

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1. Let $L$ be a linear partial differential operator defind in a domain $\mathfrak{D}$ of the $n$-dimensional Euclidean space $R^n$. For simplicity, we assume that the origin 0 of $R^n$ is contained in $\mathfrak{D}$. Denoting by $x=(x_1,\cdots,x_n)$ a point of $R^n$, we can write

$$L=L(x, D)=\sum_\alpha a_\alpha(x)D^\alpha,$$

where $\alpha$ is a sequence $(\alpha_1,\cdots,\alpha_n)$ of $n$ non-negative integers,

$$D^\alpha=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and each $a_\alpha(x)$ is a real-valued continuous function in $\mathfrak{D}$. We use a notation $|\alpha|=\alpha_1+\cdots+\alpha_n$. Put $r=|x|=(\sum_{i=1}^n x_i^2)^{1/2}$.

Friedman [3] proved the following.

Let $u(x)$ be a constant-signed solution, in $\mathfrak{D}$, of an elliptic differential equation $Lu=0$ of order $2s$. If

$$(1)\qquad \lim_{r\to 0}\frac{D^\alpha u(x)}{r^k}=0 \quad \text{for any positive integer } k,$$

where $\alpha$ is an arbitrary sequence with $|\alpha|\leqq 2s-1$, then $u(x)$ vanishes identically in $\mathfrak{D}$.

And Pederson [4] gave an improvement of this theorem. That is, he proved that, in the above theorem, the assumption (1) can be replaced by the condition that there exists a positive integer $N$ satisfying

$$\lim_{r\to 0}\frac{D^\alpha u}{r^N}=0$$

for every $\alpha(0\leqq|\alpha|\leqq 2s-1)$ and being dependent on $L$ and independent of $u$.

Now consider an elliptic system of linear partial differential equations

$$(2)\qquad \sum_{j=1}^p l_{ij}u_j=0 \quad (i=1,\cdots,p)$$

in unknown functions $u_1,\cdots,u_p$, where $l_{ij}$ is a linear partial differential operator with variable coefficients continuous in $\mathfrak{D}$. Carleman [1] proved the following.

In the case when in (2), $p=2, n=2$ and $l_{ij}(i,j=1,2)$ are of order 1, each solution $u_j$ of (2), satisfying

$$\lim_{r\to 0}\frac{u_j(x)}{r^k}=0 \quad \text{for any positive integer } k,$$

vanishes identically in $\mathfrak{D}$.

In this note, we shall give a result similar to Pederson's theorem for a system (2) under some additional conditions.

2. Before stating our result, we prove the following

**Lemma.** *Let* $L(x, D)=\sum_{\alpha} a_\alpha(x)D^\alpha$ *be an elliptic linear partial differential operator of order* $2s$. *Then there exist positive constants* $m$, $r_0$ *and* $k_0$ *such that, if* $0<r\leqq r_0$ *and* $k_0\leqq k$,

$$L(x, D)\frac{e^{\lambda r^2}}{r^{2k}}\geqq m\lambda^{2s}\frac{e^{\lambda r^2}}{r^{2(k-s)}} \qquad \text{for any } \lambda<0,$$

*where* $m$, $r_0$ *and* $k_0$ *are independent of* $\lambda$ *and, in particular,* $k_0$ *depends only on* $L(x, D)$.

*Proof.* It is easy to see that

$$(3)\qquad D^\alpha\frac{e^{\lambda r^2}}{r^{2k}} \quad \sum_{q=1}^{|\alpha|}\alpha_1!\cdots\alpha_n!\frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}\Bigg(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}x_1^{\beta_1}\cdots x_n^{\beta_n}\Bigg)$$

and

$$(4)\qquad \frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}=\Bigg(\lambda^q\frac{1}{r^{2k}}+\sum_{l=1}^{q}\lambda^{q-l}(-1)^l\binom{q}{l}k(k+1)\cdots(k+l-1)\frac{1}{r^{2(k+l)}}\Bigg)e^{\lambda r^2},$$

where $\rho=r^2$ and $\beta+2\gamma$ means the sum of vectors $(\beta_1,\cdots,\beta_n)$ and $(2\gamma_1,\cdots,2\gamma_n)$. In $\mathfrak{D}$, we consider a closed sphere with center 0 and of radius $r_1$. Since $L(x, D)$ is elliptic, we can find a positive number $c$ such that

$$\sum_{|\alpha|=2s}a_\alpha(x)x_1^{\alpha_1}\cdots x_n^{\alpha_n}\geqq cr^{2s}$$

for $r=(\sum_{i=1}^{n}x_i^2)^{1/2}\leqq r_2$. From now on, we assume $0<r\leqq r'=\mathrm{Min}\,(r_1, r_2)$ and $\lambda<0$. If $q$ is even, each term on the right hand side of (4) is positive. Hence

$$\begin{aligned}
L(x, D)\frac{e^{\lambda r^2}}{r^{2k}} &= \sum_{|\alpha|=2s}a_\alpha(x)D^\alpha\frac{e^{\lambda r^2}}{r^{2k}}+\sum_{|\alpha|\leqq 2s-1}a_\alpha(x)D^\alpha\frac{e^{\lambda r^2}}{r^{2k}}\\
&=\sum_{|\alpha|=2s}a_\alpha(x)\alpha_1!\cdots\alpha_n!\frac{d^{2s}}{d\rho^{2s}}\frac{e^{\lambda\rho}}{\rho^k}\frac{2^{2s}}{\alpha_1!\cdots\alpha_n!}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\\
&\quad+\sum_{|\alpha|=2s}a_\alpha(x)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{2s-1}\frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}\Bigg(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}x_1^{\beta_1}\cdots x_n^{\beta_n}\Bigg)\\
&\quad+\sum_{|\alpha|\leqq 2s-1}a_\alpha(x)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{|\alpha|}\frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}\Bigg(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}x_1^{\beta_1}\cdots x_n^{\beta_n}\Bigg)\\
&\geqq 2^{2s}cr^{2s}\frac{d^{2s}}{d\rho^{2s}}\frac{e^{\lambda\rho}}{\rho^k}\\
&\quad -M_1(2s+1)^{n-1}(2s)!\sum_{q=s}^{2s-1}\left|\frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}\right|(2nr)^{2q-2s}\\
&\quad -M_2\sum_{|\alpha|\leqq 2s-1}(|\alpha|+1)^{n-1}|\alpha|!\sum_{q=\left[\frac{|\alpha|}{2}\right]+1}^{|\alpha|}\left|\frac{d^q}{d\rho^q}\frac{e^{\lambda\rho}}{\rho^k}\right|(2nr)^{2q-|\alpha|},
\end{aligned}$$

where $M_1=\sup_{\substack{|\alpha|=2s\\ r\leqq r_1}}|a_\alpha(x)|$ and $M_2=\sup_{\substack{|\alpha|\leqq 2s-1\\ r\leqq r_1}}|a_\alpha(x)|$. Setting $m=2^{2s}c$ and $K_i=M_i(2s+1)^{n-1}(2s)!$ $(i=1,2)$, we get

$$L(x,D)\frac{e^{\lambda r^2}}{r^{2k}}-m\lambda^{2s}\frac{e^{\lambda r^2}}{r^{2(k-s)}}$$
$$\geqq m\sum_{l=1}^{2s}|\lambda|^{2s-l}\binom{2s}{l}k(k+1)\cdots(k+l-1)\frac{1}{r^{2(k+l-s)}}$$
$$-K_1\sum_{q=s}^{2s-1}\left(\frac{|\lambda|^q}{\rho^k}+\sum_{l'=1}^{q}|\lambda|^{q-l'}\binom{q}{l'}k(k+1)\cdots(k+l'-1)\frac{1}{r^{2(k+l'+s-q)}}\right.$$
$$-K_2\sum_{|\alpha|\leqq 2s-1}\sum_{q=\left[\frac{|\alpha|}{2}\right]+1}^{|\alpha|}\left(\frac{|\lambda|^q}{\rho^k}+\sum_{l''=1}^{q}|\lambda|^{q-l''}\binom{q}{l''}k(k+1)\cdots(k+l''-1)\right.$$
$$\left.\times\frac{1}{r^{2(k+l'')+|\alpha|-2q}}\right).$$

In the right hand side of this inequality, we compare the coefficient of each term in the first sum with that of $|\lambda|^{q-l''}$ in the third sum. If $2s-l=q-l''$, then $2(k+l-s)>2(k+l'')+|\alpha|-2q$ for any $\alpha(|\alpha|\leqq 2s-1)$ and any $q\left(\left[\frac{|\alpha|}{2}\right]+1\leqq q\leqq|\alpha|\right)$. Therefore, we can choose a sufficiently small number $r_0(<r')$ such that, when $0<r\leqq r_0$,

$$L(x,D)\frac{e^{\lambda r^2}}{r^{2k}}-m\lambda^{2s}\frac{e^{\lambda r^2}}{r^{2(k-s)}}$$
$$\geqq\frac{m}{2}\sum_{l=1}^{2s}|\lambda|^{2s-l}\binom{2s}{l}k(k+1)\cdots(k+l-1)\frac{1}{r^{2(k+l-s)}}$$
$$-K_1\sum_{q=s}^{2s-1}\left(\frac{|\lambda|^q}{\rho^k}+\sum_{l'=1}^{q}|\lambda|^{q-l'}\binom{q}{l'}k(k+1)\cdots(k+l'-1)\frac{1}{r^{2(k+l'+s-q)}}\right.\cdot$$

In the above, we compare the coefficient of $|\lambda|^{2s-l}$ of the first sum with that of the second sum. If $2s-l=q-l'$, then $2(k+l-s)=2(k+l'+s-q)$ and $l'\leqq l-1$. Hence we have the required inequality for $k\geqq k_0$ by taking $k_0$ suitably.

3. We consider linear differential operators $(l_{ij})$ $(i,j=1,\cdots,p)$ with variable coefficients defined in $\mathfrak{D}$. Every $l_{ij}$ can be expressed as follows:

$$l_{ij}=l_{ij}(x,D)=\sum_\alpha a_\alpha^{ij}(x)D^\alpha.$$

For an arbitrary real $n$-dimensional vector $\xi=(\xi_1,\cdots,\xi_n)$, there exists a one to one correspondence between operators $l_{ij}(x,D)$ and polynomials $l_{ij}(x,\xi)=\sum_\alpha a_\alpha^{ij}(x)\xi^\alpha$ in $\xi$.

Assume that there are $2p$ integers $s_1,\cdots,s_p,t_1,\cdots,t_p$ such that the order of $l_{ij}$ does not exceed $s_i+t_j$. Denote by $l'_{ij}(x,\xi)$ the sum of terms in $l_{ij}(x,\xi)$ which are exactly of order $s_i+t_j$ with respect to $\xi_1,\cdots,\xi_n$, where it is to be understood that $l_{ij}(x,\xi)\equiv 0$ if the order of $l_{ij}(x,\xi)$ in $\xi$ is less than $s_i+t_j$. The determinant $L(x,\xi)$ of the characteristic matrix

$$(5) \qquad (l'_{ij}(x,\xi))$$

of $(l_{ij})$ is a polynomial in $\xi$ which is homogeneous of degree $\sum_{i=1}^{p}(s_i+t_i)$. The system

$$\sum_{j=1}^{p} l_{ij}(x,D)u_j=0, \quad i=1,\cdots,p$$

is called elliptic, if there exist $s_i$ and $t_i$ $(i=1,\cdots,p)$ such that $L(x,\xi)$ is positive definite at every point $x$ in $\mathfrak{D}$. This definition is due to Douglis and Nirenberg [2].

We say that the elliptic system in the above sense is $(*)$-elliptic, if it satisfies the following condition:

$(*)$ When $q\neq j$, the order of

$$\sum_{i=1}^{p} L_{iq}(x,D)l_{ij}(x,D) \quad (q,j=1,\cdots,p)$$

is less than $\sum_{i=1}^{p}(s_i+t_i)$, where $L_{iq}(x,\xi)$ is the cofactor of $l'_{iq}(x,\xi)$ of the determinant of (5).

*Example*. Consider a system

$$\begin{cases} \dfrac{\partial^3 u_1}{\partial x_1^3}+\dfrac{\partial^2 u_2}{\partial x_2^2}+a_1(x)\dfrac{\partial u_1}{\partial x_1}+a_2(x)\dfrac{\partial u_1}{\partial x_2}+b_1(x)\dfrac{\partial u_2}{\partial x_1}+b_2(x)\dfrac{\partial u_2}{\partial x_1}+a_3(x)u_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +b_3(x)u_2=0, \\ -\dfrac{\partial^2 u_1}{\partial x_2^2}+\dfrac{\partial u_2}{\partial x_1}+c(x)u_1+d(x)u_2=0 \end{cases}$$

in a domain of $R^2$. Taking $s_1=2$, $s_2=1$, $t_1=1$, $t_2=0$, we see our system is $(*)$-elliptic in our sense. It is easy to verify that Carleman's system stated in §1 is also $(*)$-elliptic.

Now we can state the

**Theorem.** *Let*

$$(6) \qquad \sum_{j=1}^{p} l_{ij}(x,D)u_j=0, \qquad i=1,\cdots,p$$

*be an $(*)$-elliptic system whose coefficients have continuous derivatives of order $\sum_{i=1}^{p}(s_i+t_i)$ in $\mathfrak{D}$ and let $u_j(x)$ $(j=1,\cdots,p)$ be solutions of the above system. If each $u_i(x)$ $(1\leqq i\leqq p)$ is constant-signed, non-negative or non-positive, in $\mathfrak{D}$, then there exists a number $N$ depending only on $(l_{ij})$ such that each $u_i(x)$ vanishes identically in $\mathfrak{D}$ provided that*

$$\lim_{r\to 0}\frac{D^\alpha u_j(x)}{r^N}=0, \quad i=1,\cdots,p$$

*for every $\alpha$ $(|\alpha|\leqq\sum_{i=1}^{p}(s_i+t_i)-1)$.*

*Proof.* Operating $L_{iq}(x,D)$ to the left hand side of (6) and summing up for $i$, we have

$$(7) \qquad \sum_i L_{iq}(x,D)l_{iq}(x,D)u_q+\sum_{\substack{i,j\\ j\neq q}} L_{iq}(x,D)l_{ij}(x,D)u_j=0.$$

By the condition $(*)$, (7) can be written in the form

(8) $$L(x, D)u_q+\sum_{j=1}^{p}R_{jq}(x, D)u_j=0,$$

where $R_{jq}(x, D)$ is a linear partial differential operator whose order is less than $\sum_{i=1}^{p}(s_i+t_i)$ and $L(x, \xi)$ is the determinant of (5). Put

$$v_q=\begin{cases} u_q, & \text{if } u_q\geqq 0, \\ -u_q, & \text{if } u_q\leqq 0. \end{cases}$$

From (8), we get

$$\sum_{q=1}^{p}L(x, D)v_q=\sum_{q=1}^{p}R_q(x, D)v_q,$$

where $R_q(x, D)$ is a linear partial differential operator with order less than $\sum_{i=1}^{p}(s_i+t_i)$. We put

$$L(x, D)-R_q(x, D)=L_q(x, D)$$

and denote by $\bar{L}_q(x, D)$ the adjoint operator of $L_q(x, D)$. Since every $\bar{L}_q(x, D)$ is also an elliptic operator with coefficients continuous in $\mathfrak{D}$ and has a principal part, common with $L_q(x, D)$, of order $\sum_{i=1}^{p}(s_i+t_i)$, we can apply our lemma to $\bar{L}_q(x, D)$ and we see that there exist numbers $m_q, r_q$ and $k_0$ such that, if $\lambda<0, 0<r\leqq r_q$ and $k_0\leqq k$, it holds

$$\bar{L}_q(x, D)\frac{e^{\lambda r^2}}{r^{2k}}\geqq m_q\lambda^{\sum_{i=1}^{p}(s_i+t_i)}\frac{e^{\lambda r^2}}{r^{2k-\sum_{i=1}^{p}(s_i+t_i)}}.$$

Putting $r_0=\min_{1\leqq q\leqq p} r_q$ and $m_0=\min_{1\leqq q\leqq p} m_q$, we have

(9) $$\bar{L}_q(x, D)\frac{e^{\lambda r^2}}{r^{2k}}\geqq m_0\lambda^{\sum_{i=i}^{p}(s_i+t_i)}\frac{e^{\lambda r^2}}{r^{2k-\sum_{i=1}^{p}(s_i+t_i)}}$$

for all $q$, if $\lambda<0, 0<r\leqq r_0$ and $k_0\leqq k$. On the other hand, let $\zeta(x)$ be an infinitely differentiable function with compact carrier in $|x|<r_0$ such that $\zeta(x)=1$ in $|x|\leqq\frac{r_0}{2}$. We put $w_q(x)=\zeta(x)v_q(x)$.

By Green's formula, we get

$$\int_{\varepsilon\leqq r\leqq r_0} w_q(x)\bar{L}_q(x, D)\frac{e^{\lambda r^2}}{r^{2k}}dV_x$$
$$=\int_{\varepsilon\leqq r\leqq r_0}\frac{e^{\lambda r^2}}{r^{2k}}L_q(x, D)w_q dV_x$$
$$+\int_{r=\varepsilon}K_q\left(D^{\beta}\frac{e^{\lambda r^2}}{r^{2k}},\ D^{\gamma}w_q(x)\right)dS_x,$$

where $dV_x$ and $dS_x$ denote the volume element and the area element respectively and further $K_q\left(D^{\beta}\frac{e^{\lambda r^2}}{r^{2k}},\ D^{\gamma}w_q(x)\right)$ is a sum of products of $D^{\beta}\frac{e^{\lambda r^2}}{r^{2k}}(|\beta|\leqq\sum_{i=1}^{p}(s_i+t_i)-1)$, $D^{\gamma}w_q(x)(|\beta+\gamma|\leqq\sum_{i=1}^{p}(s_i+t_i)$, $|\gamma|\leqq\sum_{i=1}^{p}(s_i+t_i)-1)$ and bounded functions. Put $N=2(k_0+\sum_{i=1}^{p}(s_i+t_i))$. If

$$\lim_{r\to 0}\frac{D^\alpha u_q(x)}{r^{N-|\alpha|}}=0 \quad \text{for} \quad |\alpha| \leqq \sum_{i=1}^{p}(s_i+t_i)-1,$$

then, from (3),

$$\lim_{\varepsilon\to 0} K_q\left(D^\beta \frac{e^{\lambda r^2}}{r^{2k}},\ D^\gamma w_q(x)\right)_{r=\varepsilon}=0.$$

Thus we obtain

$$\int_{r\leqq r_0} \bar{L}_q(x, D)\frac{e^{\lambda r^2}}{r^{2k}} w_q(x) dV_x \leqq \int_{r\leqq r_0} \frac{e^{\lambda r^2}}{r^{2k}} L_q(x, D) w_q dV_x. \tag{10}$$

Since $L_q(x, D)w_q = L_q(x, D)v_q$ in $|x|<\frac{r_0}{2}$ and since (9) holds, the above inequality (10) implies

$$m_0 \lambda^{\sum_{i=1}^{p}(s_i+t_i)} e^{\lambda\left(\frac{r_0}{2}\right)^2} \sum_{q=1}^{p} \int_{r\leqq \frac{r_0}{2}} \frac{v_q}{r^{2k_0-\sum_{i=1}^{p}(s_i+t_i)}} dV_x$$

$$\leqq \int_{\frac{r_0}{2}\leqq r\leqq r_0} e^{\lambda r^2} \frac{\sum_{q=1}^{p} L_q(x, D) w_q}{r^{2k_0}} dV_x.$$

Dividing both sides by $m_0 \lambda^{\sum_{i=1}^{p}(s_i+t_i)} e^{\lambda\left(\frac{r_0}{2}\right)^2}$ and letting $\lambda \to -\infty$, we have

$$\int_{r\leqq \frac{r_0}{2}} \frac{\sum_{q=1}^{p} v_q}{r^{2k_0-\sum_{i=1}^{p}(s_i+t_i)}} dV_x \leqq 0.$$

Since every $v_q(x)$ is non-negative, we conclude that $v_q(x)$ vanishes in $|x|<\frac{r_0}{2}$, that is, $u_q(x)=0$ in $|x|<\frac{r_0}{2}$ $(q=1,\cdots,p)$. By a classical procedure of continuation, we see the vanishing of all $u_q(x)$ in the whole domain.

*Remark.* In the case $p=1$, the above proof gives an alternating proof of Pederson's theorem.

## References

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