137. *A Criterion for Divisors on Algebraic Varieties to be Torsion Divisors*

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The purpose of the present note is to prove the following

THEOREM. *Let $V=V_r$ be a non-singular algebraic variety of dimension $r\geqq 2$ in the N-dimensional projective space $\boldsymbol{P}_N$, and X and Y be divisors on V_r. If* $\deg X=\deg Y$ *and* $\deg(X\cdot X)=\deg(X\cdot Y)=\deg(Y\cdot Y)$, *then $X-Y$ is a torsion divisor.*

In the case where $r=2$, this was proved by Severi (cf. Zariski [5, p. 90]). (Severi [3] generalized this result in the following form for the case of arbitrary r, from which however our theorem for $r>2$ does not follow: $X-Y$ is a torsion divisor if $\deg X=\deg Y$ and $\deg(X^r)=\deg(X^{r-1}\cdot Y)=\deg(X^{r-2}\cdot Y^2)=\cdots=\deg(Y^r)$.)

We begin by the following lemma which is well-known (cf. [4, p. 214, Cor. to Th. 18]).

LEMMA 1. *Let k be a field of definition of V_r over which X and Y are rational; let H_m be a generic hypersurface over k of degree m in $\boldsymbol{P}_N$. Then the intersection product $C_m=(V\cdot H_m)_{\boldsymbol{P}_N}$*[1] *is defined and becomes a variety which is also non-singular, $(X\cdot C_m)_V$ is defined, and we have $(X\cdot C_m)_V=(X\cdot H_m)_{\boldsymbol{P}_N}$ and* $\deg(X\cdot C_m)_V=m\deg X$.

LEMMA 2. *k being as in Lemma 1, let H be a generic hyperplane over k in $\boldsymbol{P}_N$, and let $C=(V\cdot H)_{\boldsymbol{P}_N}$. Assume that the intersection product $(X\cdot Y)_V$ is defined. Then $[(X\cdot C)_V\cdot(Y\cdot C)_V]_C$ is defined, and we have $[(X\cdot C)_V\cdot(Y\cdot C)_V]_C=[(X\cdot Y)_V\cdot H]_{\boldsymbol{P}_N}$, and especially we have* $\deg[(X\cdot C)_V\cdot(Y\cdot C)_V]_C=\deg(X\cdot Y)_V$.

PROOF. Let $Y=Y_1-Y_2$, $Y_i\geqq 0$ be the reduced expression of Y. Note that $(X\cdot Y_i)_V$, $(X\cdot C)_V$, $(Y\cdot C)_V$ and $[(X\cdot Y_i)_V\cdot C]_V=[(X\cdot Y_i)_V\cdot H]_{\boldsymbol{P}_N}$ are defined. We now show that $[X\cdot(Y_i\cdot C)_V]_V$ is defined. In fact, to see this we may assume that X and Y_i are varieties defined over k. If $[X\cdot(Y_i\cdot C)_V]_V$ were not defined, then some component Z of $(Y_i\cdot C)_V$ would be contained in X, since X is a divisor. Let H be defined by the equation $u_0X_0+u_1X_1+\cdots+u_NX_N=0$, then Z is defined over the algebraic closure $\overline{k(u)}$ of $k(u)$. Take a generic point P of Z over $\overline{k(u)}$, then P would be a generic point of Y_i over k. From this and $P\in Z\subseteqq X$, would follow $Y_i\subseteqq X$. This contradicts the assump-

1) $(\ )_V$ and $(\ )_{\boldsymbol{P}_N}$ denote the intersection porducts of cycles on V and on $\boldsymbol{P}_N$ respectively.

tion that $(X\cdot Y_i)_V$ is defined, and proves our assertion. Since V and C are non-singular, and since $[(X\cdot(Y_i\cdot C)_V]_V$ and $(X\cdot C)_V$ are defined, $[(X\cdot C)_V\cdot(Y_i\cdot C)_V]_C$ is defined and we have $[(X\cdot C)_V\cdot(Y_i\cdot C)_V]_C=[X\cdot(Y_i\cdot C)_V]_V$ (cf. Weil [4, p. 214, Cor. to Th. 18]). Since $C>0$, $Y_i\geqq 0$, and since $(C\cdot Y_i)_V$, $[(C\cdot Y_i)_V\cdot X]_V$ and $(Y_i\cdot X)_V$ are defined, it follows, from Weil [4, p. 203, Cor. to Th. 10], that $[C\cdot(Y_i\cdot X)_V]_V$ is defined and $[C\cdot(Y_i\cdot X)_V]_V=[(C\cdot Y_i)_V\cdot X]_V$. From this and what we have proved above, follows $[C\cdot(Y_i\cdot X)_V]_V=[(X\cdot C)_V\cdot(Y_i\cdot C)_V]_V$, which proves $[C\cdot(Y\cdot X)_V]_V=[(X\cdot C)_V\cdot(Y\cdot C)_V]_V$.

We shall use the following properties of the virtual arithmetic genus $p_a(D)$ of a divisor D on an algebraic surface, which will follow from the properties of the characteristic linear systems:

(1) $\qquad p_a(D+E)=p_a(D)+p_a(E)+\deg(D\cdot E)_V-1$

(2) $\qquad p_a(nD)=np_a(D)+\binom{n}{2}\deg(D\cdot D)_V-n+1$

(3) $\qquad p_a(-D)=-p_a(D)+deg\,(D\cdot D)_V+2,$

where D and E are arbitrary divisors on the surface and n is an arbitrary integer $\geqq 2$ (cf. [6]).

We denote by $v(D)$ the virtual dimension of a divisor D on V_r: $v(D)=(-1)^r[p_a(V)+p_a(-D)]-1$.

LEMMA 3. *Let k and C_m be as in Lemma 1. Assume that $r=\dim V=2$. If $\deg X=\deg Y$ and $\deg(X\cdot X)_V=\deg(X\cdot Y)_V=\deg(Y\cdot Y)_V$, then we have*

(4) $\qquad p_a(-C_m-nX+nY)=p_a(-C_m)-np_a(X)+np_a(Y),$

(5) $\qquad v(C_m+nX-nY)=v(C_m)+np_a(Y)-np_a(X)$

for all integer $n\geqq 1$.

PROOF. We have $\deg(X\cdot C_m)_V=m\deg X=\deg(Y\cdot C_m)_V$ by the assumption and Lemma 1.

Case $n=1$: Making use of (1), (3) and the assumptions $\deg X=\deg Y$, $\deg(X\cdot X)=\deg(X\cdot Y)$, we can easily see $p_a(-C_m-X+Y)=p_a(-C_m)-p_a(X)+p_a(Y)$. Case of arbitrary n: Putting $A=nX$ and $B=nY$, we have $\deg A=\deg B$ and $\deg(A\cdot A)=\deg(A\cdot B)=\deg(B\cdot B)$, and so $p_a(-C_m-nX+nY)=p_a(-C_m)-p_a(nX)+p_a(nY)$. In view of (2), we have therefore

$$\begin{aligned}p_a(-C_m-nX+nY)&=p_a(-C_m)-\left[np_a(X)+\binom{n}{2}\deg(X\cdot X)-n+1\right]\\&\qquad+\left[np_a(Y)+\binom{n}{2}\deg(Y\cdot Y)-n+1\right]\\&=p_a(-C_m)-np_a(X)+np_a(Y).\end{aligned}$$

This proves Lemma 3.

PROOF OF THEOREM. Assume that $\deg X=\deg Y$ and $\deg(X\cdot X)=\deg(X\cdot Y)=\deg(Y\cdot Y)$. In the case where V is of dimension 2, Severi's proof is as follows (cf. Zariski [5, p. 90]). We can assume that $p_a(Y)\geqq p_a(X)$, without loss of generality. Fix a sufficiently large integer m which is such that $v(C_m)>0$ and $\dim|C_m+nX-nY| \geqq v(C_m+nX-nY)$ for all $n\geqq 0$. Then each linear system $|C_m+nX-nY|$ contains a positive divisor Z_n, since $\dim|C_m+nX-nY| \geqq v(C_m+nX-nY)=v(C_m)+np_a(Y)-np_a(X)>0$ by Lemma 3. The set of the Chow points $c(Z)$ of the positive divisors Z of the given degree $\deg C_m$ on V form an algebraic set W in some projective space. Since $\deg Z_n=\deg(C_m+nX-nY)=\deg C_m$, we have $c(Z_n)\in W$ $(n=0, 1, 2, \cdots)$. Therefore some component W_1 of W contains at least two points, say, $c(Z_{n_i}), c(Z_{n_j})$ $(n_i<n_j)$. (It may happen that $c(Z_{n_i})=c(Z_{n_j})$.) Thus $Z_{n_j}-Z_{n_i}$ is algebraically equivalent to zero; so that $C_m+n_jX-n_jY-(C_m+n_iX-n_iY)=(n_j-n_i)(X-Y)$ is algebraically equivalent to zero; this completes the proof of Theorem in case $r=2$.

Now assume that $r\geqq 3$ and that Theorem is proved for varieties of dimension $r-1$. Let k be an algebraically closed field of definition of V over which X and Y are rational. Let H be a generic hyperplane over k in $\boldsymbol{P}_N$, and $C=(V\cdot H)_{\boldsymbol{P}_N}$; C is a non-singular variety of dimension $r-1$. We have $\deg(X\cdot C)_V=\deg(Y\cdot C)_V$ by Lemma 1. We may assume that $(X\cdot Y)_V$ is defined, since the assumptions and the conclusion of Theorem are invariant against the linear equivalence. In view of Lemma 2, we have also $\deg[(X\cdot C)_V\cdot(Y\cdot C)_V]_C=\deg(X\cdot Y)_V$. Similarly we have $\deg[(X\cdot C)_V\cdot(X\cdot C)_V]_C=\deg(X\cdot X)_V$ and $\deg[(Y\cdot C)_V\cdot(Y\cdot C)_V]_C=\deg(Y\cdot Y)_V$. It follows that for the divisors $X_1=(X\cdot C)_V$ and $Y_1=(Y\cdot C)_V$ on C of dimension $r-1$, we have $\deg(X_1\cdot X_1)_C=\deg(X_1\cdot Y_1)_C=\deg(Y_1\cdot Y_1)_C$. It follows therefore, by the induction hypothesis, that there exists an integer $n\neq 0$ such that $nX_1-nY_1=[(nX-nY)\cdot C]_V$ is algebraically equivalent to zero on C. Since $\dim C=r-1\geqq 2$, we can conclude that $nX-nY$ is algebraically equivalent to zero on V by Matsusaka [1, p. 63, Theorem 3] and Severi [2, p. 294]; this completes our proof.

References

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