162. *An Extension of the Interpolation Theorem of Marcinkiewicz*

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§1. **Introduction.** In this paper we show that the Marcinkiewicz interpolation theorem of operators (e.g. see Zygmund [5]) holds good for Hardy class H_p or class $\mathfrak{H}_p$ introduced by Stein-Weiss [4].

H_p-class $(p>0)$ is the space of all functions analytic in the unit circle such that

$$||\varphi||_p=\lim_{r\to 1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(re^{i\theta})|^p d\theta\right\}^{1/p}$$

is finite. $\mathfrak{H}_p$-class is the space of the vectors $F(X,y)=(u(X,y), v_1(X,y),\cdots,v_n(X,y))$ whose components are all harmonic in half-space $E_{n+1}^+=\{(X,y); X\in E_n, y>0\}$[1] and satisfy the generalized Cauchy-Riemann equations,

$$\frac{\partial u}{\partial y}+\sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i}=0,\quad \frac{\partial u}{\partial x_i}=\frac{\partial v_i}{\partial y},\quad i=1,2,\cdots,n,$$

$$\frac{\partial v_i}{\partial x_j}=\frac{\partial v_j}{\partial x_i},\quad i\neq j,\quad 1\leqq i,j\leqq n,$$

and whose norm is defined by

$$||F||_p=\lim_{y\to 0}\left\{\int_{E_n}|F(X,y)|^p dx\right\}^{1/p}.$$

Let $f\in L_p(-\pi,\pi)(p\geqq 1)$ be periodic with period 2π, then its conjugate function is defined by

$$\tilde{f}(x)=\frac{1}{\pi}P.V.\int_{-\pi}^{\pi}\frac{f(y)}{2\tan(x-y)/2}dy.$$

One of its n-dimensional analogue is M. Riesz transform;

$$(Rf)(X)=((R_1f)(X),\cdots,(R_nf)(X))=\frac{1}{c_n}P.V.\int\frac{X-Y}{|X-Y|^{n+1}}f(Y)dY,$$

where $c_n=\pi^{(n+1)/2}/\Gamma((n+1)/2)$, and $f\in L_p(E_n)$.

We remark that if we put $Kf=(f+i\tilde{f})/2$ for $f\in L_p(-\pi,\pi)(p>1)$, then $Kf\in H_p$ and in particular if $f\in H_p(p\geqq 1)$, then $Kf=f$. Similarly if we put $\mathfrak{R}f=(f,Rf)=(f,R_1f,\cdots,R_nf)$ for $f\in L_p(E_n)(p>1)$, then f is a boundary function in $\mathfrak{H}_p$ and conversely if $F=(f,f_1,\cdots,f_n)$ is a boundary function in $\mathfrak{H}_p$, then $\mathfrak{R}f=F$.

§2. Let T be a quasi-linear operator from $\mathfrak{H}_p$(or H_p) to ν-

1) We denote the Euclidean space of n-dimension by E_n, its points $(x_1,\cdots,x_n)$, $(y_1,\cdots,y_n)$, etc. by X, Y, etc. and the element of volume $dx_1dx_2\cdots dx_n$ by dX.

measurable functions, that is, if TF_1 and TF_2 are defined, then $T(F_1+F_2)$ is definable and satisfies $|T(F_1+F_2)|\leqq\kappa(|TF_1|+|TF_2|)$, where κ is a constant independent on F_1 and F_2.

Theorem. *Suppose that the quasi-linear operater T satisfies*

$$\nu(\{s;\ |(TF)(s)|>t\})^{1/q_i}\leqq(M_i/t)\|F\|_{p_i},\ \text{for all } F\in\mathfrak{H}_{p_i},\quad (i=0,1)$$

where $1\leqq p_i\leqq q_i<\infty\,(i=0,1)$, $p_0\neq p_1$ and $q_0\neq q_1$. Lut us put

$$1/p=(1-\theta)/p_0+\theta/p_1 \text{ and } 1/q=(1-\theta)/q_0+\theta/q_1,\quad (0<\theta<1).$$

Then

$$\|TF\|_q\leqq\kappa(\kappa+1)AM_0^{1-\theta}M_1^{\theta}\|F\|_p,\ \text{for all } F\in\mathfrak{H}_p,$$

where A depends only on p_0, p_1, q_0, q_1 and θ, and

$$A^q=O((q_1-q)^{-1}+(q-q_0)^{-1}(p-1)^{-1}).$$

The above statements are valid for H_p-space.

Lemma 1. *Let $f\in L_p(E_n)(1<p<\infty)$, then for each $a>0$ and $r, 1\leqq r\leqq p$, the following decomposition of f is possible;*

(i) $f=u+u',\ u'=v+w,\ w=\sum_{k=1}^{\infty}w_k.$

(ii) *$u=f$, if $|f|<a$ and $u=0$, elsewhere.*

(iii) *$|v|\leqq 2^n a$, for a.e. X in E_n.*

(iv) $\displaystyle\int_{E_n}|v(X)|^s dX\leqq\int_{E_n}|u'(X)|^s dX$ *for each $s, 1\leqq s\leqq p$.*

(v) $\displaystyle\sum_{k=1}^{\infty}\int_{E_n}|w_k(X)|^s dX\leqq 2^{s+1}\int_{E_n}|u'(X)|^s dX$ *for each $s, 1\leqq s\leqq p$.*

(vi) *There exists a sequence $\{I_k\}$ of disjoint cubes such that supports of w_k are contained in I_k and*

$$\sum_{k=1}^{\infty}|I_k|\leqq\frac{1}{a^r}\int_{E_n}|u'(X)|^r dX.$$

(vii) $\displaystyle\int_{E_n}w_k(X)dX=0,\ k=1,2,\cdots.$

In the case of $L_p(-\pi,\pi)$, we decompose $f(x)$ as above for $a_0=\sup\left\{a;\pi/2\leqq a^{-r}\int|u'(x)|^r dx\right\}$ and set $f=u+u'$ for $0<a<a_0$.

In any case, we define u by (ii) and decompose $u'=f-u$ along the line in L. Hörmander [2].

Lemma 2. *For $\{w_k\}$ defined in Lemma 1, we have,*

$$\sum_{k=1}^{\infty}\int_{CE}|\mathfrak{R}w_k|dX\leqq C\sum_{k=1}^{\infty}\int_{E_n}|w_k|dX,$$

where E is the set obtained by expanding each I_k concentrically three times and CE is the complement of E and C is some constant.

Lemma 2 holds for $L_p(-\pi,\pi)$ case replacing $\mathfrak{R}w_k$ by Kw_k.

Proof of Theorem. First we consider the $\mathfrak{H}_p$-case. If $F=(f,f_1,\cdots,f_n)\in\mathfrak{H}_p$, then $F=\mathfrak{R}f$, therefore by the well-known arguments

$$\|TF\|_q^q\leqq q\int_0^{\infty}y^{q-1}\nu(|TF|>y)dy$$

$$\leqq q(3\kappa(\kappa+1))^q\Big\{\int_0^\infty y^{q-1}\nu(|T\Re u|>y)dy+\int_0^\infty y^{q-1}\nu(|T\Re v|>y)dy$$
$$+\int_0^\infty y^{q-1}\nu(|T\Re w|>y)dy\Big\}=q(3\kappa(\kappa+1))^q(I_1+I_2+I_3),\text{ say,}$$

where u, v, and w are the functions in Lemma 1 with $a=(y/b)^\lambda$, and $\nu(|TF|>y)=\nu(\{X;|(TF)(X)|>y\})$. We consider the case $1=p_0<p_1$ and $q_0<q_1$ only, the other cases are similar. I_1 may be estimated by the usual way but we must use the Calderón-Zygmund inequality $||\Re F||_p\leqq A_p||f||_p(p>1)$, where $A_p=O(p-1)^{-1}$. By (iii) in Lemma 1,

$$I_2\leqq M_1^{q_1}A_{p_1}^{q_1}\int_0^\infty y^{q-q_1-1}\Big\{\int|v(X)|^{p_1}dX\Big\}^{q_1/p_1}dy$$
$$\leqq M_1^{q_1}A_{p_1}^{q_1}2^{n(p_1-1)q_1/p_1}B^{-\lambda(p_1-1)q_1/p_1}\int_0^\infty y^{q-q_1-[(p_1-1)q_1\lambda/p_1]}\Big\{\int|v(X)|dX\Big\}^{q_1/p_1}dy.$$

Hence we get

$$I_1+I_2\leqq\Big(\frac{M_1^{q_1}A_{p_1}^{q_1}}{q_1-q}+\frac{M_1^{q_1}A_{p_1}^{q_1}2^{n(p_1-1)q_1/p_1}}{q-q_1+[(p_1-p)q_1\lambda/p_1]}\Big)\Big(\int|f|^{[(q-q_1)p_1/\lambda q_1]+p_1}dX\Big)^{q_1/p_1}.$$

$$I_3\leqq M_0^{q_0}\int_0^\infty y^{q-q_0-1}||\Re w||_{p_0}^{q_0}dy$$
$$\leqq M_0^{q_0}2^{q_0}\Big\{\int_0^\infty y^{q-q_0-1}\Big(\int_E|\Re w|\,dX\Big)^{q_0}dy+\int_0^\infty y^{q-q_0-1}\Big(\int_{CE}|\Re w|\,dX\Big)^{q_0}dy\Big\}.$$

The second term may be estimated by the well-known method applying Lemma 2.

The first term does not exceed

$$\int_0^\infty y^{q-q_0-1}|E|^{q_0/r'}\Big(\int|\Re w|^r dX\Big)^{1/r}dy,$$

where $r=(p+1)/2$ and $1/r+1/r'=1$. Using (vi) in Lemma 1 for $|E|$ and Calderón-Zygmund inequality for inner integral, above integral is not greater than

$$A_r^{q_0}2^{nq_0(r+1)/r}B^{\lambda q_0 r/r'}\int_0^\infty y^{q-q_0-1-[q_0\lambda r/r']}\Big(\int|u'|^r dX\Big)^{q_0}dy$$
$$\leqq\frac{A_r^{q_0}2^{nq_0(r+1)/r}B^{\lambda q_0 r/r'}}{q-q_0-\lambda q_0(r-1)}\Big\{\int|f|^{[(q-q_0)/q_0\lambda]+1}dX\Big\}^{q_0}.$$

Setting $\lambda=p_0(q-q_0)/q_0(p-p_0)$ and $B=M_0^\sigma M_1^\tau||f||_p^u$, σ, τ and u being some constants, we get Theorem.

In the H_p-space, we must devide the integral into $(0, y_0)$ and (y_0, ∞), where $(y_0/B)^\lambda=a_0$; we don't go into the detailed arguments.

§3. Littlewood-Paley function g^* is defined by

$$g^*(\theta,\varphi)=\Big\{\sum_{n=1}^\infty\frac{|S_n(\theta)-\sigma_n(\theta)|^2}{n}\Big\}^{1/2},$$

where $S_n(\theta)$ and $\sigma_n(\theta)$ are n-th partial sum and $(C,1)$ mean of the Fourier series of $\varphi\in H_1$. This operator is an example which is weak type $(1,1)$ for the functions in H_1-space but not in $L_1(-\pi,\pi)$ (see

E. M. Stein [3]), and which is strong type (2, 2). Another example is the operator $(T\varphi)(\theta)=S_{n(\theta)}(\theta)$, where $n(\theta)$ is any integral valued measurable function. This operator is strong type (1, 1) for $\varphi\in H_1$ when $d\nu(\theta)=d\theta/\log(|n(\theta)|+2)$ with the notation in §2 and strong type (2, 2) for $f\in L_2$. Therefore our theorem gives real proof of the Littlewood-Paley inequality $||\sup_{n\geqq 0}|S_n(\theta)/(\log(n+2))^{1/p}|\,||_p\leqq A_p||\varphi||_p$ $(1<p<2)$ (cf. H. Helson and D. Lowdenslagar [1]).

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