160. On the Existence of Local Solutions for Some Linear Partial Differential Operators

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1. *Introduction.* A problem of the existence of local solutions is stated in the following; for a linear partial differential operator $P(x, D)$ and some neighbourhood of any fixed point Ω, when we give any element f in $L^2(\Omega)$, we ask whether the equation $P(x, D)u=f$ has at least one solution u in $L^2(\Omega)$ or not, i.e. the Range of $P(x, D)$ equals to $L^2(\Omega)$? By the theorem of the range of linear transformation,[1] we shall prove the inequality of the type $||u||\leqq C||\overline{P}(x, D)u||$[2] for u belonging to $C_0^\infty(\Omega)$ which is the dense subset of the domain of $\overline{P}(x, D)$. B. Malgrange proved the existence of solutions for the operators with constant coefficients.[3] L. Hörmander proved the inequality for the operators with variable coefficients under the conditions of the principal type and that imposed on the commutator $\overline{P}_m(x, D)P_m(x, D)-P_m(x, D)\overline{P}_m(x, D)$.[4),5] M. Matsumura proved the inequality for the operators whose principal parts,[5] when they are represented products of singular integral operator[6] involving first order differential operators, have factors satisfying the condition of commutator similar to the Hörmander's.[7] As Hörmander's and Matsumura's conditions impose on the principal part of $P(x, D)$, these classes of operators contain, for example, the Laplace and wave operators but not contain the Schrödinger operator of a free particle and heat operator. Now we shall prove the inequality for a class of operators which involves not only the principal type but also the Schrödinger operator of a free particle—but this class does not contain the heat operator, for we consider the operators whose coefficients are real valued. In the case of complex valued coefficients, we shall publish later. The idea is based on Hörmander.[8] This work has been directly inspired by the uniqueness theorem obtained by Kumanogo.[9]

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1) [1] Chap. I Lem. 1.1, [7] Chap. III Th. 6 (§6).
2) $\overline{P}(x, D)$ is adjoint operator of $P(x, D)$.
3) [5].
4) $P_m(x, D)$ is the principal part of $P(x, D)$, cf. [1], [2].
5) The highest order part.
6) In the sense of Calderón and Zygmund.
7) [6].
8) [1], [2], [3].
9) [4].

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2. *Theorem and Lemmas.* Let $x=(x_1,\cdots,x_n)$ be a variable point of n-dimentional Euclidian space R^n, ω and Ω be bounded domains in R^n and u,f, etc. be without specific mention complex valued functions of x. $m=(m_1,\cdots,m_n)$ and $\alpha=(\alpha_1,\cdots,\alpha_n)$ are multi-integers. By $|\alpha:m|$ we denote $\frac{\alpha_1}{m_1}+\frac{\alpha_2}{m_2}+\cdots+\frac{\alpha_n}{m_n}$ and by m_0 we denote $\max_{1\leq j\leq n} m_j$. With these notations we shall treat the differential operator which can be written in the form

$$P(x,D)=P^0(x,D)+Q(x,D) \text{ and } P^0(x,D)=\sum_{|\alpha:m|\leq 1} a_\alpha(x)D^\alpha,$$
$$Q(x,D)=\sum_{|\alpha:m|\leq 1-\frac{1}{m_0}} a_\alpha(x)D^\alpha$$

where D^α denotes $D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ and $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$, $i=\sqrt{-1}$.

$\xi=(\xi_1,\cdots,\xi_n)$ denotes a real n-vector, ξ^α denotes $\xi_1^{\alpha_n}\cdots\xi_n^{\alpha_1}$. When we replace D^α by ξ^α in the notations of $P(x,D)$, we get $P(x,\xi)$. $P^{0(k)}(x,\xi)$ denotes $\frac{\partial P^0(x,\xi)}{\partial \xi_k}$. (u,v) denotes $\int u\bar{v}\,dx$ where $\bar{v}$ is a complex conjugate of v. $||u||$ denotes $\sqrt{(u,u)}$. For $s,t;0\leq s\leq 1, 0\leq t\leq 1, s\leq t$, $|u|_{(s)}$ and $||u||_{(t)}$ denote $\sqrt{\sum_{|\alpha:m|=s}||D^\alpha u||^2}$ and $\sqrt{\sum_{s\leq t}|u|_{(s)}^2}$ respectively. By $K(x)$ we denote $\sqrt{\sum_{j=1}^n x_j^{2m_j}}$.

Theorem. Let the coefficients of $P^0(x,D)$ *be in* C^{m_0} *and real valued and other coefficients of* $P(x,D)$ *be in* C, ω_δ *be the set* $\{x: K(x-x_0)<\delta\}$ *where* x_0 *is any fixed point in* R^n. *If there exists a constant* C[10] *such that*

(2.1) (a) $$\sum_{|\alpha:m|\leq 1-\frac{1}{m_0}}|\xi^\alpha|^2\leq C\left(\sum_{k=1}^n|P^{0(k)}(x_0,\xi)|^2+1\right)$$

for $m_0\geq 2$,

(b) *the second term in the right member of* (2.1) (a) *is ommited, hold for any* $\xi\neq 0$, *for* $m_0=1$.

Then there exist constants C_0, $\delta_0>0$ *such that if*

(2.2) $$\sum_{|\alpha:m|\leq 1-\frac{1}{m_0}}\delta^{-2(1-|\alpha:m|)}||D^\alpha u||^2\leq C_0||P(x,D)u||^2$$

if u $C_0(\omega_\delta)$ *holds.*

In the case that all m_j are equal to m_0, (2.1) (a), (b) become the equivalent condition to that of the principal type of real coefficients.[11]

10) We use the same letters C and δ for different constants so far as we do not make confusion.

11) [1] Chap. 4, [2].

To prove this theorem we use the following series of lemmas.

Lemma 1. For any ξ, the inequality

(2.3) $$|\xi^{\alpha}|^2 \leqq \{K(\xi)\}^{2|\alpha:m|} \text{ holds}$$

Lemma 2. For $s, t; 0 \leq s \leq t \leq 1$ such that $t-s=\frac{1}{m_k}$ holds for some k, there exist constants δ_0 and C such that if $0<\delta<\delta_0$ and $u \in C_0^\infty(\Omega_\delta)$, the inequality

(2.4) $|u|_{(s)}^2 \leq C\delta^{2(t-s)}|u|_{(t)}^2$ holds, where Ω_δ is the set $\{x; K(x)<\delta\}$.

Lemma 3. Let $p(x, D)$ and $q(x, D)$ be two differential operators $\sum_{|\alpha:m|=1} a_\alpha(x)D^\alpha$ and $\sum_{|\alpha:m|=1-\frac{1}{m_k}} b_\alpha D^\alpha$ and respectively. If the coefficients of $p(x, D)$ and $q(x, D)$ are in C^{m_0}, the equality

$$(p(x, D)v, q(x, D)u)$$
$$=(\bar{q}(x, D)v, \bar{p}(x, D)u)+\sum_{j=1} \sum_{|\alpha:m|\leq 1-\frac{1}{m_j}} \sum_{|\beta:m|=1-\frac{1}{m_k}} (C_{\alpha\beta}D^\alpha v, D^\beta u)$$

where $C_{\alpha\beta}$ are in C, and $\bar{p}(x, D), \bar{q}(x, D)$ denote

$$\bar{p}(x, D)=\sum_{|\alpha:m|=1} \overline{a_\alpha(x)}D^\alpha, \quad \bar{q}(x, D)=\sum_{|\beta:m|=1-\frac{1}{m_k}} \overline{b_\beta(x)}D^\beta$$

respectively and u, v in C_0^∞.

Lemma 4. Let the inequality (2.2) be hold for $P(x, D)$ and $P(x, D)$ have the coefficients in C. Then for any other operator with the same part $P^0(x, D)$ the inequality (2.2) also holds.

Here we remark only that particularly in the case of all m_j being equal to m_0, these lemmas are equivalent to those of Hörmander.[12]

3. *Proof of theorem.* In what follows we shorten the notations; $P^0(x, D)=P^0$ and $P^{0(k)}(x, D)=P^{0(k)}$.

Noting that Leibniz formula gives

$$P^0(x, D)(ix_k u)=P^{0(k)}u+ix_k P^0 u$$

we get $(P^{0(k)}u, P^{0(k)}u)=(P^0 u, P^{0(k)}u)-(ix_k P^0 u, P^{0(k)}u)$. Without loss of generality let x_0 be 0. Using Schwartz inequality and $|x_k|<l_k$ in ω_δ, we get for $u \in C_0(\omega_\delta)$

$$||P^{0(k)}u||^2=Re(P^0(ix_k u), P^{0(k)}u)+l_k||P^0 u|| \, ||P^{0(k)}u||.$$

Next we shall estimate the first term in the right member. Using lemma 3, we can write it in the following;

$$Re(P^0(ix_k u), P^{0(k)}u)$$
$$=Re(\bar{P}^{0(k)}(ix_k u), \bar{P}^0 u)+Re\sum_{j=1}^{n} \sum_{|\alpha:m|\leqq 1-\frac{1}{m_j}}^{n} \sum_{|\beta:m|=1-\frac{1}{m_k}} (c_{\alpha\beta}D^\alpha(ix_k u), D^\beta u)$$
$$=Re(ix_k\bar{P}^{0(k)}u, \bar{P}^0 u)+Re(\bar{P}^{0(kk)}u, \bar{P}^0 u)$$
$$+Re\sum_{j=1}^{n} \sum_{|\alpha:m|\leqq 1-\frac{1}{m_j}} \sum_{|\beta:m|=1-\frac{1}{m_k}} (c_{\alpha\beta}ix_k D^\alpha u, D^\beta u)$$
$$+Re\sum_{j=1} \sum_{|\alpha:m|\leqq 1-\frac{1}{m_j}-\frac{1}{m_k}} \sum_{|\beta:m|=1-\frac{1}{m_k}} (c_{\alpha\beta}D^\alpha u, D^\beta u),$$

12) [2], but in Lemma 3 C^{m_0} is replaced by C^1 in Hörmander's.

where $P^{0(kk)}$ denotes $\frac{\partial}{\partial \xi_k} P^{0(k)}(x, \xi)$. Therefore we get by choosing constants properly

$$\|P^{0(k)}u\| \leqq l_k \|\overline{P}^{0(k)}u\| \, \|\overline{P}^0 u\| + \|\overline{P}^{0(kk)}u\| \, \|\overline{P}^0 u\| + l_k \|P^0 u\| \, \|P^{0(k)}u\|$$

$$+ C l_k \sum_{j=1}^{n} \sum_{|\alpha: m| \leqq 1-\frac{1}{m_j}} \sum_{|\beta: m|=1-\frac{1}{m_k}} \|D^\alpha u\| \, \|D^\beta u\| + C \sum_{j=1}^{n} \sum_{|\alpha: m| \leqq 1-\frac{1}{m_j}-\frac{1}{m_k}} \sum_{|\beta: m|=1-\frac{1}{m_k}}$$

$$\|D^\alpha u\| \, \|D^\beta u\| \leq l_k C |u|_{(1-\frac{1}{m_k})} \|\overline{P}^0 u\| + C |u|_{(1-\frac{2}{m_k})} \|\overline{P}^0 u\| + C l_k \|P^0 u\| \, |u|_{(1-\frac{1}{m_k})}$$

$$+ C l_k |u|_{(1-\frac{1}{m_k})} \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} + C |u|_{(1-\frac{1}{m_k})} \sum_{j=1} \|u\|_{(1-\frac{1}{m_k}-\frac{1}{m_j})}$$

$$\leq C l_k \Big(\sum_{k=1}^{n} |u|_{(1-\frac{1}{m_k})} \|\overline{P}^0 u\| + C \big(\sum_{k=1}^{n} |u|_{(1-\frac{1}{m_k}-\frac{1}{m_j})} \Big) \|\overline{P}^0 u\| + C l_k \sum_{j=1}^{n} |u|_{(1-\frac{1}{m_j})}$$

$$+ C l_k \Big(\sum_{j=1}^{n} |u|_{(1-\frac{1}{m_j})} \Big) \Big(\sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big) + C \Big(\sum_{j=1}^{n} |u|_{(1-\frac{1}{m_j})} \Big) \Big(\sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_k}-\frac{1}{m_j})} \Big)$$

$$\geq C \Big(l_k \Big(\|\overline{P}^0 u\| + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big) \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} + \Big(\|\overline{P}^0 u\| + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big)$$

$$\times \Big(\sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_k}-\frac{1}{m_j})} \Big) + C l_k \|P^0 u\| \, \|u\|_{(1-\frac{1}{m_k})}$$

$$= C \Big(\|\overline{P}^0 u\| + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big) + \Big(l_k \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j}-\frac{1}{m_k})} \Big)$$

$$+ C l_k \|P^0 u\| \, \|u\|_{(1-\frac{1}{m_k})}$$

and we choose δ such that for $l=(l_1, \cdots, l_k, \cdots, l_n)$ $\delta^2 = \{K(l)\}^2$ and $\delta<1$ hold, and then by lemma 1 $l_k \leqq \{K(l)\}^{\frac{2}{m_k}} = \delta^{\frac{2}{m_k}}$ is satisfied, and we can continue

$$= C \delta^{\frac{2}{m_k}} \Big(\|\overline{P}^0 u\| + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big) \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} + C \delta^{\frac{2}{m_k}} \|\overline{P}^0_u\| \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})}$$

if $u \in C_0^\infty(\omega_\delta)$.

Summing up each $\|P^{0(k)}u\|$, we obtain for other constant C

$$\sum_{k=1}^{n} \|P^{0(k)}u\| \leq C \delta^{\frac{1}{m_0}} \Big(\|\overline{P}^0 u\| + \|P^0 u\| + \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \Big) \sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})}.$$

As $m_0 = \max_{1 \leqq j \leqq n} m_j$, we can affirm the inequality

$$\sum_{j=1}^{n} \|u\|_{(1-\frac{1}{m_j})} \leq C' \|u\|_{(1-\frac{1}{m_0})}.$$

Therefore we get

$$\sum_{j=1}^{n} \|P^{0(k)}u\| \leq C \delta^{\frac{2}{m_0}} (\|\overline{P}^0 u\| + \|P^0 u\| + \|u\|_{(1-\frac{1}{m_0})} \|u\|_{(1-\frac{1}{m_0})}$$

To prove (2.2), it only remains to show that

$$\text{(3.1)} \qquad \|u\|_{(1-\frac{1}{m_0})} \leq C \sum_{k=1}^{n} \|P^{0(k)}(x, D)u\| \qquad u \in C_0^\infty(\omega_{\tilde{\delta}}) \tilde{\delta} < \delta.$$

For combined with above to inequalies, this gives

$$(1 - C \delta^{\frac{2}{m_0}} \|u\|_{(1-\frac{1}{m_0})} \leq C \delta^{\frac{2}{m_0}} (\|P^0(x, D)u\| + \|\overline{P}^0(x, D)u\|)$$

and δ chosen properly we can take $1 - C\delta^{\frac{2}{m_0}} > 0$.

By Lemma 2 if $u \in C_0^\infty(\omega_{\tilde{\delta}})$ $\tilde{\delta} < \delta$

$$\sum_{|\beta:m|=s}\|D^\beta u\|^2\le C\tilde{\delta}^{2\left(1-\frac{1}{m_0}-s\right)}\sum_{|\alpha:m|=1-\frac{1}{m_0}}\|D^\alpha u\|^2,\quad\text{hence}$$

$$\sum_{|\beta:m|\le 1-\frac{1}{m_0}}\tilde{\delta}^{-2\left(1-\frac{1}{m_0}-|\beta:m|\right)}\|D^\beta u\|\le C\,|u|_{\left(1-\frac{1}{m_0}\right)}\le C\|u\|_{\left(1-\frac{1}{m_0}\right)}$$

holds. Hence we get $\sum_{|\alpha:m|\le 1-\frac{1}{m_0}}\tilde{\delta}^{-2\left(1-|\alpha:m|+\frac{1}{m_0}\right)}\|D^\alpha u\|=C\tilde{\delta}^{\frac{2}{m_0}}(\|P^0u\|+\|P^0u\|)$. Multiplying $\tilde{\delta}^{\frac{2}{m_0}}$ both sides, being the coefficients of P^0 real, and applying lemma 4 for P^0 and $P=P^0+Q$, we affirm the inequality (2.2) for P.

So we are going to prove the inequality (3.1). By the assumption of the Theorem (2.1) (a) and Parseval formula and Lemma 2 we get

$$\|u\|^2_{\left(1-\frac{1}{m_0}\right)}\le C\left(\sum_{k=1}^n\|P^{0(k)}(0,D)u\|^2+\|u\|^2\right)$$
$$\le C\left(\sum^n\|P^{0(k)}\ (o,D)u\|^2+\tilde{\delta}^{\left(1-\frac{1}{m_0}\right)}\ \|u\|^2_{\left(1-\frac{1}{m_0}\right)}\right).$$

Choosing $\tilde{\delta}$ properly small, we get

(3.2) (a) $$\|u\|^2_{\left(1-\frac{1}{m_0}\right)}\le C\left(\sum_{k=1}^n\|P^{0(k)}(o,D)u\|^2\right).$$

On the other hand if $\tilde{\delta}$ is small, for $u\in C_0^\infty(\omega_{\tilde{\delta}})$.

$$\|P^{0(k)}(o,D)u-P^{0(k)}(x,D)u\|^2\le C'\tilde{\delta}\sum_{|\alpha:m|\le 1-\frac{1}{m_k}}\|D^\alpha u\|^2\le C''\delta\sum_{|\alpha:m|\le 1-\frac{1}{m_0}}\|D^\alpha u\|^2$$

holds.

$$\sum_{k=1}^n\|P^{0(k)}(0,D)u\|\le\sum_{k=1}^n(\|P^{0(k)}(x,D)u\|+\|P^{0(k)}(o,D)u-P^{0(k)}(x,D)u\|)$$
$$=\sum_{k=1}^n\|P^{0(k)}(x,D)u\|+C''\tilde{\delta}\|u\|_{\left(1-\frac{1}{m_0}\right)}.$$

For combined this and (3.2) (a) we get (3.1). For $m_0=1$, similar calculi lead to (3.1) immediately. This completes the proof.

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