159. On Distributions and Spaces of Sequences. IV

On Generalized Multiplication of Distributions

By Tadashige ISHIHARA and Hideo YAMAGATA
Department of Mathematics, Toyonaka, Osaka University
(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1962)

1. **Introduction.** In the previous article [2] published under the same title, we considered the equivalent classes $\mathfrak{c}(T|\tau_1|\widetilde{T}_\alpha)$, $\mathfrak{c}(T|\tau_1|\infty_\beta)$ and the ranges of product $\boldsymbol{R}[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\mathfrak{D}'}$, $\boldsymbol{R}[\mathfrak{c}(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )]_{\mathfrak{D}'}$.

In this article we investigate detail relations between the topologies τ_1, τ_2 and the ranges $\boldsymbol{R}[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\mathfrak{D}'}$, $\boldsymbol{R}[\tau(T|\tau_1|\ )\circ\mathfrak{c}(S|\tau_2|\ )]_{\mathfrak{D}'}$. We also give here full explanation to our considerations which are discussed in [1] about Theorem given by L. Schwartz. We add here also some corrections to the errors found in the previous articles [I] and [2].

2. **Notations and Definitions.** We consider the set of all sequences $\{\varphi_n\}$ of functions $\varphi_n\in(\mathcal{E})$. In this set we introduce the following relations:

(1) $\{\psi_n\}=\{\varphi_n\}\Longleftrightarrow\psi_n=\varphi_n$ for all n,

(2) $\{\psi_n\}\pm\{\varphi_n\}=\{\psi_n\pm\varphi_n\}$,

and construct the linear space $\boldsymbol{Q}$.

Let $\widetilde{\boldsymbol{Q}}_\tau$ denote the subspace of all convergent sequences in τ topology (on $(\mathcal{E})$), where τ is a topology which is finer than $\tau_{\mathfrak{D}'}$ and is compatible with the linear operations in $(\mathcal{E})$.

Let $\boldsymbol{O}_\tau$ denote the set of all sequences which converge to zero in τ topology. Let Q_τ denote the set of classes such that $Q_\tau\equiv\boldsymbol{Q}/O_\tau=\{\mathfrak{c}(\ |\tau|\widetilde{T}_\alpha),\ \mathfrak{c}(\ |\tau|\infty_\beta)\}$.

Let $\widetilde{Q}_\tau$ be the set of all convergent classes, i.e.

$$\widetilde{Q}_\tau\equiv\widetilde{\boldsymbol{Q}}_\tau/\boldsymbol{O}_\tau=\{\mathfrak{c}(\ |\tau|\widetilde{T}_\alpha)\}.$$

We consider the set of all convergent (in $\tau_{\mathfrak{D}'}$) sequences $\{\varphi_n\}$, $\varphi_n\in(\mathcal{E})$. In this set, we introduce the above relations (1), (2), and construct the linear space $\boldsymbol{Q}^{\mathfrak{D}'}$. Let $\widetilde{\boldsymbol{Q}}_\tau^{\mathfrak{D}'}$ denote the subspace of all convergent sequences in τ topology which is contained in $\boldsymbol{Q}^{D'}$. Let $Q_\tau^{\mathfrak{D}'}$ be the set of all classes; $Q_\tau^{\mathfrak{D}'}\equiv\boldsymbol{Q}^{\mathfrak{D}'}/\boldsymbol{O}_\tau=\{\mathfrak{c}(T|\tau|\widetilde{T}_\alpha),\ \mathfrak{c}(T|\tau|\infty_\beta)\}$, where $\varphi_n\in\mathfrak{c}(T|\tau|\widetilde{T}_\alpha)$ means φ_n converge to T in $(\mathfrak{D}')$, and φ_n converge to $\widetilde{T}_\alpha$ in τ. Let $\widetilde{Q}_\tau^{\mathfrak{D}'}$ denote the set of all convergent classes of $Q_\tau^{D'}$ i.e. $\widetilde{Q}_\tau^{\mathfrak{D}'}\equiv\widetilde{\boldsymbol{Q}}_\tau^{\mathfrak{D}'}/\boldsymbol{O}_\tau=\{\mathfrak{c}(T|\tau|\widetilde{T}_\alpha)\}$.

Let P_τ be the natural mapping from $\boldsymbol{Q}$ to Q_τ or $\boldsymbol{Q}^{\mathfrak{D}'}$ to $Q_\tau^{\mathfrak{D}'}$.

Let $\mathfrak{c}(T|\tau_1|\ )$ be the element of $Q_{\tau_1}^{\mathfrak{D}'}$, and let $\mathfrak{c}(S|\tau_2|\ )$ be the element of $Q_{\tau_2}^{\mathfrak{D}'}$.

Definition.

$\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )=\{\{\psi_n\};\ \psi_n=\varphi_n^T\cdot\varphi_n^S,\ \{\varphi_n^T\}\in\mathfrak{c}(T|\tau_1|\ ),\ \{\varphi_n^S\}\in\mathfrak{c}(S|\tau_2|\ )\}$,

$\boldsymbol{R}[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}=P_{\tau_3}\{\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )\}$.

$\widetilde{\boldsymbol{R}}[\mathfrak{c}(T|\tau_1|\ )\cdot\mathfrak{c}(S|\tau_2|\ )]_{\tau_3}$ *is the set of all convergent classes in* τ_3 *topology.*

The ordinary multiplication is defined between $\alpha\in(\mathcal{E})$ and $T\in(\mathfrak{D}')$, and is contained in our definition as follows;

$\alpha\leftrightarrow\mathfrak{c}(\alpha|\mathcal{E}|\alpha),\ T\leftrightarrow\mathfrak{c}(T|\mathfrak{D}'|T)$ and

$\alpha T\leftrightarrow\boldsymbol{R}[\mathfrak{c}(\alpha|\mathcal{E}|\alpha)\cdot\mathfrak{c}(T|\mathfrak{D}'|T)]_{D'}=\mathfrak{c}(\alpha T|\mathfrak{D}'|\alpha T)$.

Remark 1. *We add the following two cases to Definition* 2 *in* [2],

(1) $n_1(m)=\infty,\ n_2(m)\neq\infty$; namely $\{\psi_m\}=\{[\lim_{n\to\infty}\varphi_n^T\cdot\varphi_{n_2(m)}^S]\}$,

(2) $n_1(m)\neq\infty,\ n_2(m)=\infty$; namely $\{\psi_m\}=\{[\lim_{n\to\infty}\varphi_{n_1(m)}^T\cdot\varphi_n^S]\}$.

Remark 2. *We add "and is finer than $\tau_{\mathfrak{D}'}$" after the word $\mathfrak{D}$ in the 4th line* p. 329 [1].

3. **Range of product.** Let $\{W(0)\}$ denote the neighbourhoods of zero in $(\mathfrak{D}')$. Let τ_0 be the topology defined by the linear shift of the following neighbourhoods of zero:

$$V_\varepsilon(0)\cap W(0)=\{\varphi;\ \sup|e^{-1/x^2}\varphi|<\varepsilon,\ \varphi\in W(0)\}.$$

Let $\tilde{\tau}_0^k$ be the topology defined by the linear shift of the following neighbourhoods of zero $\{V_\varepsilon(0)\cap U_{\varepsilon'}^k(0)\cap W(0)\}$ in which $U_{\varepsilon'}^k(0)=\{\varphi;\ \sup|x^k\varphi|<\varepsilon'\}$, where k is a fixed positive integer. When we can take an arbitrary fixed positive number as k, we denote this topology $\tilde{\tau}_0$ in generalization.

Lemma 1. τ_0 *and* $\tilde{\tau}_0$ *are compatible with the linear operations in* $(\mathcal{E})$.

Proof. The mappings $\varphi\to\alpha\varphi$, $(\varphi,\psi)\to\varphi+\psi$ are continuous, because $\alpha U_\alpha^\varepsilon(\varphi)\subseteq U_\varepsilon(\alpha\varphi)$ and $U_{\varepsilon/2}(\varphi)+U_{\varepsilon/2}(\psi)\subseteq U_\varepsilon(\varphi+\psi)$. Hence these topologies are compatible with the linear operations.

According to [1] we can construct the classes $\mathfrak{c}(\delta|\tau_0|\tilde{\delta}_1)$, $\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_2)$.

Lemma 2. *The elements of the converging class $\mathfrak{c}(\delta|\tau_0|\tilde{\delta}_1)$ are the sequences $\{\varphi_n\}$ such that $\lim_{n\to\infty}\varphi_n=0$ uniformly for $|x|>\varepsilon$, where ε is an arbitrary positive number.*

Proof. According to the definition of τ_0, $\{\varphi_n\}$ is an uniformly convergent sequence in $|x|>\varepsilon$. Hence for $|x|>\varepsilon$, $(\lim_{n\to\infty}\varphi_n)\in C$. On the other hand $\lim_{n\to\infty}\varphi_n=\delta$ in $(\mathfrak{D}')$. Hence for $|x|>\varepsilon$ $\lim_{n\to\infty}\varphi_n=0$ uniformly.

Lemma 3. *The element of $\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta})$ is contained in a certain convergent class $\mathfrak{c}(\delta|\tau_0|\tilde{\delta})$.*

Proof. If $\{\varphi_n^\delta\}$ converge in $\tilde{\tau}_0$, then $\{\varphi_n^\delta\}$ converge in τ_0. Hence, any $\{\varphi_n\}\in\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta})$ is contained in a certain convergent class $\mathfrak{c}(\delta|\tau_0|\ )$.

We denote this class by $\mathfrak{c}(\delta|\tau_0|\tilde{\delta})$. Since $\tilde{\tau}_0$ is stronger than τ_0, $\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta})\subset\mathfrak{c}(\delta|\tau_0|\tilde{\delta})$.

Lemma 4. *For $\{\varphi_n\}\in\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_1)$, $|x^k\varphi_n|$ has an upper bound which is independent from n.*

It is easily seen from the definition of $\tilde{\tau}_0$.

Theorem 1. *The elements of $\boldsymbol{R}[\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_1)\cdot\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_2)]_{\mathfrak{D}'}$ are sequences $\{\varphi_n\}$ such that $\langle\{\varphi_n\}\cdot\varphi\rangle=\sum_{i=0}^{2k-1}C_i\varphi^{(i)}(0)$ for $\varphi\in(\mathfrak{D})$, where C_i are determined as follows;*

(1) *C_i is a finite number,*

or (2) *C_i is $\pm\infty$,*

or (3) *C_i is determined in the meaning of* (1) *or* (2), *if we select the suitable subsequences from the given sequence $\{\varphi_n\}$.*

Proof. Let's decompose $\varphi\in(\mathfrak{D})$ to the following sum;

$$\varphi=\{(a_0+a_1x+(a_2\,x^2/2!)+\cdots+(a_{2k-1}x^{2k-1})/(2k-1)!)\}\beta(x)+x^{2k}\psi(x),$$

where $\beta(x)\in(\mathfrak{D})$ and satisfy two conditions (1) $0\leqq\beta(x)\leqq1$ and (2) $\beta(x)=\begin{cases}1 & \text{for } x\in(\text{carrier of }\varphi),\\ 0 & \text{for } x\notin U(\text{carrier of }\varphi),\end{cases}$

where U(carrier of φ) is a suitable neighbourhood of the carrier of φ and ψ is a function of the space $(\mathfrak{D})$.

Let's take two series $\{\varphi_n^{(1)}\}\in\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_1)$, $\{\varphi_n^{(2)}\}\in\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_2)$ and construct a series $\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\}$. Then we obtain the following formula; $\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\}\cdot\varphi\rangle$
$=\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},\ \{(a_0+a_1x+a_2x^2/2!+\cdots+a_{2k-1}x^{2k-1}/(2k-1!)\beta(x)+x^{2k}\psi(x)\}\rangle$

$$=\sum_{i=0}^{2k-1}a_i\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},\ x^i\,\beta(x)/i!\rangle+\langle\{x^k\varphi_n^{(1)}\cdot x^k\varphi_n^{(2)}\},\ \psi(x)\rangle.$$

Let M denote $\sup\{|x|;\ x\in(\text{carrier of }\psi(x))\}$.
For any $\varepsilon',\varepsilon''>0$, there exists $N>0$ such that for $n>N$ the inequality $|x^{2k}(\varphi_n^{(1)}\cdot\varphi_n^{(2)})|<\varepsilon'^2e^{2/x^2}M^{2k}$ is satisfied for $|x|\geqq\varepsilon''>0$.

Let B be a common upper bound of $\{\ |x^{2k}\varphi_n^{(1)}\cdot\varphi_n^{(2)}\psi|\ \}$. Then, for any $\varepsilon>0$, there exists $N>0$ such that

$$\left|\int x^{2k}(\varphi_n^{(1)}\cdot\varphi_n^{(2)})\psi dx\right|\leqq\int_{|x|<\varepsilon/4B}|x^{2k}(\varphi_n^{(1)}\cdot\varphi_n^{(2)})\psi|\,dx+\int_{|x|\geqq\varepsilon/4B}|x^{2k}(\varphi_n^{(1)}\cdot\varphi_n^{(2)})\psi|\,dx$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon\quad\text{for}\quad n>N.$$

Hence $\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},\varphi\rangle=\sum_{i=0}^{2k-1}a_i\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},\ x^i\beta(x)/i!\rangle$
$=\sum_{i=0}^{2k-1}\varphi^{(i)}(0)\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},x^i\beta(x)/i!\rangle$. $\langle\{\varphi_n^{(1)}\cdot\varphi_n^{(2)}\},x^i\beta(x)/i!\rangle$ are determined such that one of the three cases occurs.

Remark 3. We write the contents of (2) and (4) in Theorem 1, in correction of Theorem 3 p. 336 in the previous article [2].

We correct similarly p. 334 Theorem 1 and p. 336 Theorem 2.

Theorem 2. *The elements of* $\tilde{\boldsymbol{R}}[\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_1)\cdot\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_2)]_{\mathfrak{D}'}$ *are expressed as the formula* $\sum_{i=0}^{2k-1}C_i\delta^{(i)}$, *where* C_i *are finite numbers.*

Proof. It is evident from the conclusion of Theorem 2 and the definition of $\tilde{\boldsymbol{R}}[\mathfrak{c}(\delta(\tilde{\tau}_0|\tilde{\delta}_1)\cdot\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta}_2)]_{D'}$.

Example. Let's construct the following two sequences $\{\varphi_n^{(1)}(x)\}$, $\{\varphi_n^{(2)}(x)\}$:

$$\varphi_n^{(1)}(x)=\begin{cases} n & \text{for } -1/2n+1/n^2<x<1/2n-1/n^2, \\ 0 & \text{for } x-1/2n \text{ and for } x>1/2n, \end{cases}$$

$\varphi_n^{(1)}(x)\in(\mathfrak{D})$ and $0\leq\varphi_n^{(1)}(x)\leq n$.

$$\varphi_n^{(2)}(x)=\begin{cases} 2n & \text{for } 1/2n\leq x<1/n \\ a+(-1)^n b & \text{for } 0\leq x<1/2n-1/n^2 \\ 0 & \text{for } x<-1/n^2 \text{ and for } x>1/n+1/n^2, \end{cases}$$

$\varphi_n^{(2)}(x)\in(\mathfrak{D})$ and $0\leq\varphi_n^{(2)}(x)\leq\max(2n,|a+(-1)^n b|)$.

Then, $\lim_{n\to\infty}\varphi_n^{(1)}(x)=\delta$ and $\lim_{n\to\infty}\varphi_n^{(2)}(x)=\delta$. But $\varphi_n^{(1)}(x)\cdot\varphi_n^{(2)}(x)$ does not converge.

Let's consider similarly the convergent classes $\mathfrak{c}(1/x|\tau_0|\widetilde{1/x})$, $\mathfrak{c}(1/x|\tilde{\tau}_0|\widetilde{1/x})$, $\mathfrak{c}(\delta'|\tau_0|\tilde{\delta}')$ and $\mathfrak{c}(\delta'|\tilde{\tau}_0^k|\tilde{\delta}')$ $(k\geq 2)$.
Then, we can easily prove the following five lemmas by the definition of the topologies τ_0 and $\tilde{\tau}_0$.

Lemma 5. *The elements of* $\mathfrak{c}(1/x|\tau_0|\widetilde{1/x})$ *are the sequences* $\{\varphi_n\}$ *such that* $\lim_{n\to\infty}|\varphi_n-1/x|=0$ *uniformly for* $|x|>\varepsilon$ *where* ε *is an arbitrary positive number.*

Lemma 6. *The elements of* $\mathfrak{c}(\delta'|\tau_0|\delta')$ *are the sequences* $\{\varphi_n\}$ *such that* $\lim_{n\to\infty}\varphi_n=0$ *uniformly for* $|x|>\varepsilon$, *where* ε *is an arbitrary positive number.*

Lemma 7. *The elements of* $\mathfrak{c}(1/x|\tilde{\tau}_0|\widetilde{1/x})$ *are contained in a certain converging class* $\mathfrak{c}(1/x|\tau_0|\widetilde{1/x})$, *and the elements of* $\mathfrak{c}(\delta'|\tilde{\tau}_0^k|\tilde{\delta}')$ *are contained in a certain converging class* $\mathfrak{c}(\delta'|\tau_0|\tilde{\delta}')$.

Lemma 8. *For the element* $\{\varphi_n\}\in\mathfrak{c}(1/x|\tilde{\tau}_0|\widetilde{1/x})$, $|x^k\varphi_n|$ *has an upper bound which is independent from* n.

Lemma 9. *For the element* $\{\varphi_n\}\in\mathfrak{c}(\delta'|\tilde{\tau}_0^k|\tilde{\delta}')$, $|x^k\varphi_n|$ *has an upper bound which is independent from* n.

Theorem 3. *The element of* $\boldsymbol{R}[\mathfrak{c}(1/x|\tilde{\tau}_0|\widetilde{1/x})\cdot\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta})]_{\mathfrak{D}'}$, $\boldsymbol{R}[\mathfrak{c}(\delta'|\tilde{\tau}_0^k|\tilde{\delta}')\cdot\mathfrak{c}(1/x|\tilde{\tau}_0|\widetilde{1/x})]_{\mathfrak{D}'}$ *and* $\boldsymbol{R}[\mathfrak{c}(\delta'|\tilde{\tau}_0^k|\tilde{\delta}')\cdot\mathfrak{c}(\delta|\tilde{\tau}_0|\tilde{\delta})]_{\mathfrak{D}'}$ *are sequences* $\{\varphi_n\}$ *such that* $\langle\{\varphi_n\},\varphi\rangle=\sum_{i=0}^{2k-1}C_i\varphi^{(i)}(0)$ *for* $\varphi\in(\mathfrak{D})$, *where* C_i *are determined as follows;*

(1) C_i *is a finite number,*

or (2) C_i *is* $\pm\infty$,

or (3) C_i *is determined in the meaning of* (1) *or* (2), *if we select the suitable subsequence from the given sequence* $\{\varphi_n\}$.

The proof is same as the proof of Theorem 2.

4. On impossibility of multiplication investigated by L. Schwartz. L. Schwartz showed the following Theorem [10]: Let E be a vector space on real numbers, one of whose subspace is the space F of all continuous functions of one real variable. The multiplication which satisfies the following conditions cannot exist.

(1) coincides with usual multiplication on F,

(2) $1 \in E$ where $1 \cdot e = e$ for any $e \in E$,

(3) bilinear associative operation,

(4) $x^{-1} \in E$ where $x^{-1} \cdot x = 1$,

(5) $\delta \in E$ where $x \cdot \delta = 0$, $\delta \neq 0$.

If we try to define the multiplication of improper function, accepting strictly the meaning of multiplication in this theorem, then we see that we cannot construct it consistently.

So, we modify the meaning of multiplication, as already discussed in this article. We accept at first that the distribution is a class $\mathfrak{c}(T|\tau|\ ) \in Q_\tau^{\mathfrak{D}'}$. Namely we interprete that the distribution is a class of sequences which also depend on the topology τ. Moreover at the correspondence between T and $\mathfrak{c}(T|\tau|\ )$, we sometimes use the different topologies for the same T. For example, if T is a multiplicant or a multiplier, we use $\mathfrak{c}(T|\mathcal{E}|\ )$, and if T is a multiple, we use $\mathfrak{c}(T|\mathfrak{D}'|\ )$.

L. Schwartz also use this interpretation and define the multiplication. In fact, in multiplication he sometimes uses the function f which corresponds to distribution T_f. For example, $\langle 1 \cdot T, \varphi \rangle = \langle T, 1 \cdot \varphi \rangle = \langle T, \varphi \rangle$. On the other hand, in the following formula $x \cdot pf\, 1/x = 1$, he uses the equality by the meaning of ($\mathfrak{D}'$) topology and uses T_1.

According to our interpretation, the same element 1 has different topologies in these two cases. Namely, 1 in the first case is $\mathfrak{c}(1|\tau_\varepsilon|\tilde{1})$ and 1 in the second case is $\mathfrak{c}(1|\tau_{D'}|\tilde{1})$.

If we admit these modified meanings of multiplication, we may be able to define consistently the multiplication of improper function. In fact, the multiplication in page 331 [1] satisfies the conditions (1), (2), (3), (4), and (5) in the above modified sense.

Remark 4. *We want to correct the topology* (τ_0) *in page* 331 [1] *used in* (1), (2), (3) *as follows*: "τ_0 *is the discrete topology*".

Now, we show that the condition (3) is satisfied in the following form, too.

(a) The multiplication defined in this article is the bilinear operation in the following meaning: For arbitrary $\{\varphi_n\} \in \{\mathfrak{c}(T_1|\tau_1|\ ) \cdot (\mathfrak{c}(T_2|\tau_2|\ ) + \mathfrak{c}(T_3|\tau_3|\ ))\}$ and $\{\psi_n\} \in \{\mathfrak{c}(T_1|\tau_1|\ ) \cdot \mathfrak{c}(T_2|\tau_2|\ )\} + \{\mathfrak{c}(T_1|\tau_1|\ ) \cdot \mathfrak{c}(T_3|\tau_3|\ )\}$,

$\{\varphi_n\} \equiv \{\psi_n\}$ $\{\bmod\ c(0|\tau_1|\ )\cdot c(T_2|\tau_2|\ )$ and $\bmod\ c(0|\tau_1|\ )\cdot c(T_3|\tau_3|\ )\}$.

Furthermore, for arbitrary $\{\varphi_n\} \in \{(c(T_2|\tau_2|\ )+c(T_3|\tau_3|\ ))\cdot c(T_1|\tau_1|\ )\}$ and $\{\psi_n\} \in \{c(T_2|\tau_2|\ )\cdot c(T_1|\tau_1|\ )\}+\{c(T_3|\tau_3|\ )\cdot c(T_1|\tau_1|\ )\}$, $\{\varphi_n\} \equiv \{\psi_n\}$ $\{\bmod\ c(T_2|\tau_2|\ )\cdot c(0|\tau_1|\ )$ and $\bmod\ c(T_3|\tau_3|\ )\cdot c(0|\tau_1|\ )\}$.

(b) The multiplication in this article satisfies the condition of associativity in the following meaning: $\{c(T|\tau_1|\ )\cdot c(S|\tau_2|\ )\}\cdot c(Q|\tau_3|\ ) = c(T|\tau_1|\ )\cdot\{c(S|\tau_2|\ )\cdot c(Q|\tau_3|\ )\} = c(T|\tau_1|\ )\cdot c(S|\tau_2|\ )\cdot c(Q|\tau_3|\ )$. We can discuss similarly about commutativity.

References

[1] T. Ishihara and H. Yamagata: On distributions and spaces of sequences, I. General theory, Proc. Japan Acad., **38**, 327–332 (1952).

[2] T. Ishihara and H. Yamagata: On distributions and spaces of sequences, II. On the multiplications of improper functions in quantum field theory, Proc. Japan Acad., **38**, 333–338 (1962).

[3] T. Ishihara and H. Yamagata: On distributions and spaces of sequences, III. On powers of the distributions, Proc. Japan Acad., **38**, 711–714 (1962).

[4] L. Schwartz: Théorie des Distributions, I, II. Hermann (1950), (1951).

[5] И. М. Гельфанд и Г. Е. Шилов: Обобщенные Функции и Действия над Ними. 1, 2, 3. (1959).

[6] H. König: Neue Begrundung der Theorie der "Distributionen" von L. Schwartz. Math. Nachr., **9**, 129 (1953).

[7] H. König: Multiplikation von Distributionen. I. Math. Annalen, **128**, 420 (1954).

[8] E. Stueckelberg and A. Petermann: La normalisation des constantes dans la theorie des quanta, Helv. Phys. Acta., **26**, 499 (1953).

[9] W. Güttinger: Product of improper operators and the renormalization problem of quantum field theory, Prog. of Theor. Phys. **13**, 612 (1955).

[10] L. Schwartz: Sur l'impossibilité de la multiplication des distributions, Comptes Rendus Acad. Sci. (Paris), **239**, 847 (1954).