157. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. V

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1962)

In this paper, for the function $S(\lambda)$ defined in the statement of Theorem 1 we shall discuss its behaviour in the exterior of an appropriately large circle with center at the origin and shall show the orthogonality between its ordinary part and its two principal parts.

Theorem 13. Let $S(\lambda)$ and $\{\lambda_\nu\}$ be the same notations as those defined in the statement of Theorem 1, and let $M_S(\rho, 0)$ denote the maximum of the modulus $|S(\lambda)|$ as λ ranges over the points of the circle $|\lambda|=\rho$ satisfying $\sup_\nu|\lambda_\nu|<\rho<\infty$. Then a necessary and sufficient condition that the ordinary part $R(\lambda)$ of $S(\lambda)$ be a polynomial in λ of degree less than or equal to d is that there exist a positive number K and an appropriately large number σ such that the inequality

$$M_S(r, 0) \geqq Kr^d \tag{12}$$

holds for every r satisfying $\sup_\nu|\lambda_\nu|<\sigma<r<\infty$.

Proof. If we put

$$\begin{cases} a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} S(\rho e^{it})\cos nt\,dt \\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} S(\rho e^{it})\sin nt\,dt \end{cases} \qquad (n=0,1,2,\cdots)$$

where ρ is an arbitrarily given number such that $\sup_\nu|\lambda_\nu|<\rho<\infty$, then, as shown in Theorem 7, for any κ subject to the condition $0<\kappa<1$ we have

$$S(\rho e^{i\theta}/\kappa)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)(e^{i\theta}/\kappa)^n+\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)(\kappa/e^{i\theta})^n$$

(θ: variable),

where the series on the right-hand side converges absolutely and uniformly, and hence

$$\frac{1}{2\pi}\int_0^{2\pi}|S(\rho e^{i\theta}/\kappa)|^2\,d\theta=\frac{1}{4}\Big(|a_0|^2+\sum_{n=1}^{\infty}|a_n-ib_n|^2\frac{1}{\kappa^{2n}}+\sum_{n=1}^{\infty}|a_n+ib_n|^2\kappa^{2n}\Big).$$

The final equality here implies that

$$\frac{1}{2}\Big(|a_0|^2+\sum_{n=1}^{\infty}|a_n-ib_n|^2\frac{1}{\kappa^{2n}}+\sum_{n=1}^{\infty}|a_n+ib_n|^2\kappa^{2n}\Big)^{\frac{1}{2}}\leqq M_S(\rho/\kappa, 0),$$

so that

$$|a_n - ib_n| \leqq 2M_S(\rho/\kappa, 0)\kappa^n \quad (n=1, 2, 3, \cdots).$$

On the other hand, if we suppose that there exist a positive number K and an appropriately large number σ such that the inequality (12) holds for every r satisfying $\sup_\nu |\lambda_\nu| < \sigma < r < \infty$, the inequality

$$M_S(\sigma/\kappa, 0) \leqq K(\sigma/\kappa)^d$$

holds for every κ with $0<\kappa<1$. In consequence, by taking for ρ the above-mentioned σ we have

$$|a_n - ib_n| \leqq 2K\sigma^d \kappa^{n-d} \quad (0<\kappa<1),$$

which shows that $a_n - ib_n = 0$ for $n>d$. Since, in addition,

$$R^{(n)}(0) = \frac{n!(a_n - ib_n)}{2\rho^n} \quad (n=0, 1, 2, \cdots)$$

as shown in the course of the proof of Theorem 6, we obtain the relation $R^{(n)}(0)=0$ holding for every positive integer n larger than d. Thus the condition stated in the present theorem is sufficient.

Conversely we now suppose that $R(\lambda)$ is a polynomial in λ of degree less than or equal to d. Then

$$\begin{aligned} a_n - ib_n &= \frac{2\rho^n R^{(n)}(0)}{n!} \\ &= 0 \qquad\qquad (n>d), \end{aligned}$$

so that

$$M_S(\rho/\kappa, 0) \leqq \frac{|a_0|}{2} + \frac{1}{2}\sum_{n\leqq d} |a_n - ib_n| \frac{1}{\kappa^n} + \frac{1}{2}\sum_{n=1}^{\infty} |a_n + ib_n| \kappa^n$$
$$(0<\kappa<1, \ \sup_\nu |\lambda_\nu| < \rho < \infty),$$

as will be found immediately from the expansion of $S(\rho e^{i\theta}/\kappa)$.

Since, on the other hand,

$$|a_n \pm ib_n| \leqq \frac{1}{\pi}\int_0^{2\pi} |S(\rho e^{it})|\, dt \leqq 2M_S(\rho, 0) < \infty \quad (n=0, 1, 2, \cdots),$$

it follows from the just established inequality that

$$\begin{aligned} M_S(\rho/\kappa, 0) &\leqq \frac{1-\kappa^{d+1}}{(1-\kappa)\kappa^d} M_S(\rho, 0) + \frac{\kappa}{1-\kappa} M_S(\rho, 0) \\ &\leqq \frac{1}{(1-\mu)\kappa^d} M_S(\rho, 0) \end{aligned}$$

for an arbitrarily given positive number μ less than unity and for every κ with $0<\kappa\leqq\mu$. Putting now

$$\frac{1}{(1-\mu)\kappa^d} M_S(\rho, 0) = K$$

for an arbitrarily prescribed number ρ with $\sup_\nu |\lambda_\nu| < \rho < \infty$, we obtain therefore

$$M_S(\rho/\kappa, 0) \leqq K(\rho/\kappa)^d$$

for every positive number κ less than or equal to μ. This result shows that the condition under consideration is necessary.

Thus the theorem has been proved.

Theorem 14. Let $S(\lambda)$, $R(\lambda)$, and $M_S(\rho, 0)$ be the same notations as those used in the preceding theorem respectively. Then, in order that $R(\lambda)$ should be a transcendental integral function of order $d>0$, it is necessary and sufficient that

$$\overline{\lim_{\rho\to\infty}} \frac{\log\log M_S(\rho, 0)}{\log \rho} = d, \quad d>0.$$

Proof. Since, by definition, $R(\lambda)$ is either a polynomial (inclusive of a constant) or a transcendental integral function, to prove this theorem it is sufficient to show that

$$\overline{\lim_{\rho\to\infty}} \frac{\log\log M_S(\rho, 0)}{\log \rho} = \overline{\lim_{\rho\to\infty}} \frac{\log\log M_R(\rho, 0)}{\log \rho}, \tag{13}$$

where $M_R(\rho, 0)$ denotes the maximum of the modulus $|R(\lambda)|$ on the circle $|\lambda|=\rho$.

Let $\Phi(\lambda)$ and $\Psi(\lambda)$ denote the first and second principal parts of $S(\lambda)$ respectively. Since, as shown by the relations (3) and (4) in the course of the proof of Theorem 4, the relations

$$R(z) = \frac{1}{2}R(0) + \frac{1}{4\pi}\int_0^{2\pi} S(\lambda)\frac{\lambda+z}{\lambda-z}dt,$$

$$\bar{\Phi}(\lambda\bar{\lambda}/\bar{z}) + \bar{\Psi}(\lambda\bar{\lambda}/\bar{z}) + \frac{1}{2}\bar{R}(0) = \frac{1}{4\pi}\int_0^{2\pi} \bar{S}(\lambda)\frac{\lambda+z}{\lambda-z}dt$$

$$(\lambda=\rho e^{it},\ \sup_\nu |\lambda_\nu| < \rho < \infty)$$

hold for any point $z=re^{i\theta}$ in the interior of the circle $|\lambda|=\rho$, we have

$$\bar{S}(\rho e^{i\theta}/\kappa) - \bar{R}(\rho e^{i\theta}/\kappa) = -\frac{1}{2}\bar{R}(0) + \frac{1}{4\pi}\int_0^{2\pi}\bar{S}(\rho e^{it})\frac{e^{it}+\kappa e^{i\theta}}{e^{it}-\kappa e^{i\theta}}dt \quad \left(\kappa=\frac{r}{\rho}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\bar{S}(\rho e^{it})\frac{\kappa e^{i\theta}}{e^{it}-\kappa e^{i\theta}}dt$$

and hence

$$|S(\rho e^{i\theta}/\kappa) - R(\rho e^{i\theta}/\kappa)| \leqq \frac{\kappa}{1-\kappa}M_S(\rho, 0) \quad (0<\kappa<1).$$

Putting $\frac{\kappa}{1-\kappa}M_S(\rho, 0)=\varepsilon(\kappa)$ for simplicity, the last relation leads us to the chain of inequalities

$$M_S(\rho/\kappa, 0) - \varepsilon(\kappa) \leqq M_R(\rho/\kappa, 0) \leqq M_S(\rho/\kappa, 0) + \varepsilon(\kappa), \tag{14}$$

where $M_R(\rho/\kappa, 0)$ is a monotone-increasing function of ρ/κ unless $R(\lambda)$ is a constant. Furthermore we have for any sufficiently small positive κ

$$\log\log[M_S(\rho/\kappa, 0)-\varepsilon(\kappa)] = \log\left[\log M_S(\rho/\kappa, 0) + \log\left(1-\frac{\varepsilon(\kappa)}{M_S(\rho/\kappa, 0)}\right)\right]$$

$$> \log\left[\log M_S(\rho/\kappa, 0) - \frac{\alpha\varepsilon(\kappa)}{M_S(\rho/\kappa, 0)}\right]$$

$$\left(\alpha = \frac{M_S(\rho/\kappa, 0)}{M_S(\rho/\kappa, 0)-\varepsilon(\kappa)}\right)$$

$$> \log\log M_S(\rho/\kappa,0) - \frac{\alpha\beta\varepsilon(\kappa)}{M_S(\rho/\kappa,0)\log M_S(\rho/\kappa,0)}$$
$$\left(\beta = \frac{M_S(\rho/\kappa,0)\log M_S(\rho/\kappa,0)}{M_S(\rho/\kappa,0)\log M_S(\rho/\kappa,0)-\alpha\varepsilon(\kappa)}\right)$$

and

$$\log\log[M_S(\rho/\kappa,0)+\varepsilon(\kappa)] = \log\left[\log M_S(\rho/\kappa,0)+\log\left(1+\frac{\varepsilon(\kappa)}{M_S(\rho/\kappa,0)}\right)\right]$$
$$< \log\left[\log M_S(\rho/\kappa,0)+\frac{\varepsilon(\kappa)}{M_S(\rho/\kappa,0)}\right]$$
$$< \log\log M_S(\rho/\kappa,0)+\frac{\varepsilon(\kappa)}{M_S(\rho/\kappa,0)\log M_S(\rho/\kappa,0)}.$$

Letting $\kappa \to +0$, the final two inequalities established above and (14) permit us to conclude that the relation (13) holds true, as we wished to prove.

The proof of the present theorem is thus complete.

Definition. Let $L_2[0,2\pi]$ be the class of all complex-valued Lebesgue-measurable functions, $f(\rho e^{i\theta})$, defined almost everywhere in the closed interval $[0,2\pi]$ of θ for which the Lebesgue integral $\int_0^{2\pi}|f(\rho e^{i\theta})|^2 d\theta$ exists, two functions $f(\rho e^{i\theta})$ and $g(\rho e^{i\theta})$ of this class being considered as identical if and only if $f(\rho e^{i\theta})=g(\rho e^{i\theta})$ almost everywhere in the interval $[0,2\pi]$ of θ; and let the inner product $(f(\rho e^{i\theta}), g(\rho e^{i\theta}))$ be defined by the equation $(f(\rho e^{i\theta}), g(\rho e^{i\theta})) = \int_0^{2\pi} f(\rho e^{i\theta})\bar{g}(\rho e^{i\theta})d\theta$. If $f(\rho e^{i\theta})$ and $g(\rho e^{i\theta})$ belong to $L_2[0,2\pi]$ and $(f(\rho e^{i\theta}), g(\rho e^{i\theta}))=0$, then $f(\lambda)$ and $g(\lambda)$ are said to be orthogonal on the circle $|\lambda|=\rho$.

Theorem 15. Let $S(\lambda)$, $\{\lambda_\nu\}$, and $R(\lambda)$ be the same notations as before, and let $\Phi(\lambda)$ and $\Psi(\lambda)$ be the first and second principal parts of $S(\lambda)$ respectively. Then $R(\lambda)$ is orthogonal to $\Phi(\lambda)$ and to $\Psi(\lambda)$ on any circle $|\lambda|=\rho$ with $\sup_\nu|\lambda_\nu|<\rho<\infty$, and $\Psi(\lambda)$ is also orthogonal to $\bar{\Phi}(\lambda)$ on any such circle. Moreover the relation

$$(15)\qquad \int_0^{2\pi}|S(\rho e^{i\theta})|^2 d\theta = \int_0^{2\pi}|\Phi(\rho e^{i\theta})+\Psi(\rho e^{i\theta})|^2 d\theta + \int_0^{2\pi}|R(\rho e^{i\theta})|^2 d\theta$$

is valid for any ρ subject to the condition $\sup_\nu|\lambda_\nu|<\rho<\infty$.

Proof. As shown by (7) in Theorem 6, the equality

$$R(\rho e^{i\theta}/\kappa) = \frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)(e^{i\theta}/\kappa)^n \quad (\theta\text{: variable})$$

holds for any κ with $0<\kappa<\infty$ and for any ρ with $\sup_\nu|\lambda_\nu|<\rho<\infty$ and the series on the right-hand side is absolutely and uniformly convergent; and moreover, as shown in the course of the proof of Theorem 1, the equality

$$\Phi(\rho e^{i\theta}/\kappa)=\sum_{\alpha=1}^{m}\sum_{\nu}c_{\alpha}^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_{\nu}\right)^{-\alpha}\quad(\theta\text{: variable})$$

holds for any κ with $0<\kappa<1$ and for any ρ with $\sup_{\nu}|\lambda_{\nu}|<\rho<\infty$ and the series on the right-hand side is absolutely and uniformly convergent by virtue of the hypothesis on $c_{\alpha}^{(\nu)}$. On the other hand, as indicated in Remark of Theorem 8, we have the equality

$$\Psi(\rho e^{i\theta}/\kappa)=\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\left(\frac{\kappa}{e^{i\theta}}\right)^{n}-\sum_{\alpha=1}^{m}\sum_{\nu}c_{\alpha}^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_{\nu}\right)^{-\alpha}\quad(\theta\text{: variable})$$

holding for any κ with $0<\kappa<1$ and for any ρ with $\sup_{\nu}|\lambda_{\nu}|<\rho<\infty$, where the series on the right-hand side converges absolutely and uniformly. As we already pointed out, however, $\Psi(\lambda)$ vanishes necessarily if all the accumulation points of $\{\lambda_{\nu}\}$ form a countable set. In addition,

$$(16)\qquad \sum_{\alpha=1}^{m}\sum_{\nu}c_{\alpha}^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}-\lambda_{\nu}\right)^{-\alpha}=\sum_{\alpha=1}^{m}\sum_{\nu}c_{\alpha}^{(\nu)}\left(\frac{\rho e^{i\theta}}{\kappa}\right)^{-\alpha}\times\left[1+\sum_{s=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+s-1)}{s!}\lambda_{\nu}^{s}\left(\frac{\rho e^{i\theta}}{\kappa}\right)^{-s}\right]$$

and

$$\sum_{\alpha=1}^{m}\sum_{\nu}|c_{\alpha}^{(\nu)}|\left(\frac{\rho}{\kappa}\right)^{-\alpha}\left[1+\sum_{s=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+s-1)}{s!}|\lambda_{\nu}^{s}|\left(\frac{\rho}{\kappa}\right)^{-s}\right]$$
$$=\sum_{\alpha=1}^{m}\sum_{\nu}|c_{\alpha}^{(\nu)}|\frac{\left(\frac{\kappa}{\rho}\right)^{\alpha}}{\left(1-\frac{\kappa|\lambda_{\nu}|}{\rho}\right)^{\alpha}}\quad(0<\kappa<1,\ \sup_{\nu}|\lambda_{\nu}|<\rho<\infty)$$
$$<\frac{1}{(1-\kappa)^{m}}\sum_{\alpha=1}^{m}\sum_{\nu}|c_{\alpha}^{(\nu)}|\left(\frac{\kappa}{\rho}\right)^{\alpha}.$$

Consequently the expansion of the right-hand member in (16) is absolutely and uniformly convergent; and the above-mentioned expansion of each of the functions $R(\rho e^{i\theta}/\kappa)$, $\Phi(\rho e^{i\theta}/\kappa)$, $\Psi(\rho e^{i\theta}/\kappa)$, and its absolute and uniform convergence enable us to show that

$$(\Phi(\rho e^{i\theta}/\kappa),R(\rho e^{i\theta}/\kappa))=(\Psi(\rho e^{i\theta}/\kappa),R(\rho e^{i\theta}/\kappa))=(\Psi(\rho e^{i\theta}/\kappa),\bar{\Phi}(\rho e^{i\theta}/\kappa))=0.$$

Since this chain of relations holds for any κ with $0<\kappa<1$ and for any ρ with $\sup_{\nu}|\lambda_{\nu}|<\rho<\infty$, the former half of the theorem has thus been proved. Moreover, by applying this result we obtain

$$||S(\rho e^{i\theta})||^2=||\Phi(\rho e^{i\theta})+\Psi(\rho e^{i\theta})+R(\rho e^{i\theta})||^2$$
$$=||\Phi(\rho e^{i\theta})+\Psi(\rho e^{i\theta})||^2+||R(\rho e^{i\theta})||^2,$$

that is, the relation (15).

With these results, the proof of the theorem is complete.