20. *On the Composition of a Summable Function and a Bounded Function*

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1. Introduction. The main purpose of this paper is to argue the generalized harmonic analysis of a function of composition type in the Weyl space. Let $f(x)$ be a bounded measurable function and $K(x)$ be a summable function on $(-\infty, \infty)$. Let us consider the composition of f and K:

$$g(x)=\int_{-\infty}^{\infty} K(x-y)f(y)dy=K*f. \tag{1.1}$$

Let us denote by $s(u, x)$ the Fourier-Wiener transform of $f(x+t)$ where we take "t" as variable:

$$\begin{aligned} s(u, x)=&\operatorname*{l.i.m.}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]\frac{f(x+t)e^{-iut}}{-it}dt \\ &+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x+t)\frac{e^{-iut}-1}{-it}dt. \end{aligned} \tag{1.2}$$

Let us introduce the norm which was firstly defined by H. Weyl [1] in the study of almost periodic functions. It concerns with measurable and integrable function in any finite interval and such that

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f(t)|^2 dt<\infty. \tag{1.3}$$

By $f\sim g$ we mean that we have

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f(t)-g(t)|^2 dt=0. \tag{1.4}$$

For the sake of simplicity we use the notation

$$||f||_p=\left(\int_{-\infty}^{\infty} |f(t)|^p dt\right)^{1/p} \quad (p>0). \tag{1.5}$$

Then the main result of this paper is as follows:

Theorem 1. Let $f(x)$ and $g(x)$ be bounded measurable functions on $(-\infty, \infty)$. Let $K(x)$ be a measurable function of the class $L_1(-\infty, \infty)$. Let us denote by $s(u, x)$ and $t(u, x)$ the Fourier-Wiener transform of $f(x+t)$ and $g(x+t)$ respectively. Let us put

$$I(\varepsilon, x)=\frac{1}{\varepsilon}||\{t(u+\varepsilon, x)-t(u-\varepsilon, x)\}-k(u)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}||_2^2 \tag{1.6}$$

where $k(u)$ is the Fourier transform of $K(t)$. Then under the supplementary condition

(1.7) $$\int_{-\infty}^{\infty} |t|^{1/2}|K(t)|\,dt<\infty,$$

for that there exists the following relation between f and g

(1.8) $$g\sim K*f,$$

it is necessary and sufficient that

(1.9) $$\sup_{-\infty<x<\infty} I(\varepsilon, x)<\infty \text{ and } \overline{\lim_{\varepsilon\to 0}}\sup_{x} I(\varepsilon, x)=0$$

are satisfied.

This result corresponds to the A. Beurling theorem [4], and is analogous form of the N. Wiener theorem [12, p. 170]. The method of proof can be done by running on his lines, but for the sake of uniformity it is not easy. We also supply the one-sided Wiener formula which has established in the previous paper [9]. In the last we shall define spectrum of bounded function in some sense.

2. Theorems on composition. Proof of Theorem 1. The proof of necessity. By the one-sided Wiener formula we may assume that $g=K*f$ without loss of generality. We get uniformly as for x in $(-\infty, \infty)$:

$$\begin{aligned} &k(u)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}\\ &\quad=\int_{-\infty}^{\infty} K(\xi)e^{-iu\xi}d\xi \operatorname*{l.i.m.}_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(t+x)\frac{2\sin \varepsilon t}{t}e^{-iut}dt\\ &\quad=\int_{-\infty}^{\infty} K(\xi)d\xi \operatorname*{l.i.m.}_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(s-\xi+x)\frac{2\sin \varepsilon(s-\xi)}{s-\xi}e^{-ius}ds.\end{aligned}$$

Because we have

$$\left\|\left[\int_{-A}^{-A+\xi}+\int_{A}^{A+\xi}\right]f(s-\xi+x)\frac{2\sin\varepsilon(s-\xi)}{s-\xi}e^{-ius}ds\right\|_2^2\leqq 32\pi\varepsilon \sup_{-\infty<x<\infty}|f(x)|^2.$$

Let us put

$$F(u, \xi, x, A)=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(s-\xi+x)\frac{2\sin\varepsilon(s-\xi)}{s-\xi}e^{-ius}ds.$$

Then we get

$$||F(u, \xi, x, A)||_2^2\leqq 8\pi \sup|f|^2.$$

Let us also put

$$F(u, \xi, x)=\operatorname*{l.i.m.}_{A\to\infty} F(u, \xi, x, A).$$

Then we get

$$||F(u, \xi, x)-F(u, \xi, x, A)||_2^2\leqq \sup|f|^2\varepsilon\left[\int_{-\infty}^{-\varepsilon(A+\xi)}+\int_{\varepsilon(A-\xi)}^{\infty}\right]\frac{4\sin^2\xi}{\xi^2}d\xi$$

and we get over any finite range of ξ

$$\frac{1}{\varepsilon}||F(u, \xi, x)-F(u, \xi, x, A)||_2^2\to 0 \quad (A\to\infty)$$

uniformly as for x in $(-\infty, \infty)$. We also have

$$|F(u, \xi, x, A)|\leqq \frac{2\sqrt{2}A}{\sqrt{\pi}}\sup|f|.$$

From these estimations we get

$$
\begin{aligned}
&k(u)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\} \\
(2.1)\qquad &=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} e^{-ius}ds\int_{-\infty}^{\infty} f(s-\xi+x)\frac{2\sin\varepsilon(s-\xi)}{s-\xi}K(\xi)d\xi,
\end{aligned}
$$

uniformly as for x in $(-\infty, \infty)$.

Next we get

$$
\begin{aligned}
&t(u+\varepsilon, x)-t(u-\varepsilon, x) \\
(2.2)\qquad &=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\int_{-\infty}^{\infty}K(\xi)f(t+x-\xi)d\xi.
\end{aligned}
$$

From (2.1) and (2.2) we get

$$
\begin{aligned}
&\{t(u+\varepsilon, x)-t(u-\varepsilon, x)\}-k(u)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\} \\
(2.3)\qquad &=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}e^{-iut}dt\int_{-\infty}^{\infty}2f(t+x-\xi)\left[\frac{\sin\varepsilon t}{t}-\frac{\sin\varepsilon(t-\xi)}{t-\xi}\right]K(\xi)d\xi.
\end{aligned}
$$

Here we borrow the lemma due to N. Wiener [12, p. 157]:

Lemma 1. We have

$$
(2.4)\qquad \left|\frac{\sin\varepsilon t}{t}-\frac{\sin\varepsilon(t-\xi)}{t-\xi}\right|\leqq\frac{16\,\varepsilon|\xi|}{|t|+|\xi|}.
$$

Substituting (2.4) into (2.3) we get

$$
I(\varepsilon, x)\leqq 32\,\varepsilon\sup|f|^2\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{|\xi|}{|t|+|\xi|}\,|K(\xi)|\,d\xi\right|^2dt.
$$

By the Schwartz inequality we get

$$
\begin{aligned}
\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{|\xi|}{|t|+|\xi|}\,|K(\xi)|\,d\xi\right|^2dt&\leqq\left(\int_{-\infty}^{\infty}|\xi|^{1/2}|K(\xi)|\,d\xi\right)^2\left(\int_{-\infty}^{\infty}\frac{|\xi|}{(|t|+|\xi|)^2}dt\right) \\
&\leqq\pi\left(\int_{-\infty}^{\infty}|\xi|^{1/2}|K(\xi)|\,d\xi\right)^2.
\end{aligned}
$$

Therefore we obtain

$$
I(\varepsilon, x)\leqq 32\,\pi\,\varepsilon\sup|f|^2\left(\int_{-\infty}^{\infty}|\xi|^{1/2}|K(\xi)|\,d\xi\right)^2.
$$

Thus the necessity is proved.

Proof of sufficiency. This is easily obtained by the one-sided Wiener formula. We omit details.

We prove a theorem on dilatation:

Theorem 2. Let $f(x)$ be a bounded measurable function. Let us denote by $s(u, x)$ the Fourier-Wiener transform of $f(x+t)$. Let us put

$$
\begin{aligned}
(2.5)\qquad J(x, y, \varepsilon)=\frac{1}{\varepsilon}\|&\{s(u+\varepsilon, x+y)-s(u-\varepsilon, x+y)\} \\
&-e^{iuy}\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}\|_2^2.
\end{aligned}
$$

Then we get over any finite range of y

$$
(2.6)\qquad \sup_{-\infty<x<\infty}J(x, y, \varepsilon)<\infty \quad\text{and}\quad \overline{\lim_{\varepsilon\to 0}}\sup_{x}J(x, y, \varepsilon)=0.
$$

This is an analogous form of the N. Wiener theorem [12, p. 158].

Proof of Theorem 2. We get

$$
\{s(u+\varepsilon, x+y)-s(u-\varepsilon, x+y)\}-e^{iuy}\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}
$$

$$=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(t+x)\left[\frac{2\sin\varepsilon(t-y)}{t-y}-\frac{2\sin\varepsilon t}{t}\right]e^{-iu(t-y)}dt.$$

Applying Lemma 1 to the above formula we get

$$J(x,y,\varepsilon)\leqq 16^2\varepsilon\int_{-\infty}^{\infty}|f(t+x)|^2\frac{y^2}{(|t|+|y|)^2}dt\leqq 16^2\pi^2|y|\varepsilon\cdot\sup|f|^2.$$

By the aid of this theorem we can prove the second theorem on composition.

Theorem 3. Under the same assumption and notation as Theorem 1 except the supplementary condition (1.7), we get

$$\overline{\lim_{\varepsilon\to 0}}\sup_{-\infty<x<\infty}\frac{1}{\varepsilon}||t(u+\varepsilon,x)-t(u-\varepsilon,x)||_2^2 \tag{2.7}$$
$$=\overline{\lim_{\varepsilon\to 0}}\sup_{-\infty<x<\infty}\frac{1}{\varepsilon}||k(u)\{s(u+\varepsilon)-s(u-\varepsilon)\}||_2^2.$$

Proof of Theorem 3. Let us put for $N>1, K_N=K$ if $|K|<N$ and $K_N=(\mathrm{sign}\,K)N$ if $|K|\geqq N$. Then we have $K_N\in L_1\cap L_2(-\infty,\infty)$ and $||K-K_N||_1\to 0(N\to\infty)$. Let us denote by $t_N(u,x)$ the Fourier-Wiener transform of $(K_N*f)(t+x)$. Then we get

$$t_N(u+\varepsilon,x)-t_N(u-\varepsilon,x)=\int_{-\infty}^{\infty}K_N(\xi)\{s(u+\varepsilon,x-\xi)-s(u-\varepsilon,x-\xi)\}d\xi.$$

Here let us put

$$\Delta(u,x-\xi,\varepsilon)=\{s(u+\varepsilon,x-\xi)-s(u-\varepsilon,x-\xi)\}-e^{-iu\xi}\{s(u+\varepsilon,x)-s(u-\varepsilon,x)\}$$

then we get

$$||t_N(u+\varepsilon,x)-t_N(u-\varepsilon,x)||_2^2$$
$$=||k_N(u)\{s(u+\varepsilon,x)-s(u-\varepsilon,x)\}+\int_{-\infty}^{\infty}K_N(\xi)\Delta(u,x-\xi,\varepsilon)d\xi||_2^2$$

where $k_N(u)$ is the Fourier transform of $K_N(t)$.

We have

$$\overline{\lim_{\varepsilon\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{\varepsilon}\left|\left|\int_{-\infty}^{\infty}K_N(\xi)\Delta(u,x-\xi,\varepsilon)d\xi\right|\right|_2^2=0. \tag{2.8}$$

Because if we write $|K_N\Delta|=|K_N|^{1/2}|\,|K_N|^{1/2}\Delta|$ and apply the Schwartz inequality we get for any finite $L>0$

$$\frac{1}{\varepsilon}\left|\left|\int_{-L}^{L}K_N(\xi)\Delta(u,x-\xi,\varepsilon)d\xi\right|\right|_2^2\leqq\int_{-L}^{L}|K_N(\xi)|d\xi\sup_{-L<\xi<L}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}|\Delta(u,x-\xi,\varepsilon)|^2du.$$

Applying Theorem 2 we get

$$\sup_x\frac{1}{\varepsilon}\left|\left|\int_{-L}^{L}K_N(\xi)\Delta(u,x-\xi,\varepsilon)d\xi\right|\right|_2^2\leqq\sup_{-L<\xi<L}\frac{1}{\varepsilon}||\Delta(u,x-\xi,\varepsilon)||_2^2\int_{-L}^{L}|K_N|d\xi\to 0$$

as $\varepsilon\to 0$.

On the other hand we get

$$\frac{1}{\varepsilon}\left|\left|\int_{L}^{\infty}K_N(\xi)\Delta(u,x-\xi,\varepsilon)d\xi\right|\right|_2^2$$
$$\leqq\frac{1}{\varepsilon}\left|\left|\int_{L}^{\infty}K_N(\xi)\{s(u+\varepsilon,x-\xi)-s(u-\varepsilon,x-\xi)\}d\xi\right|\right|_2^2+$$

$$+\frac{1}{\varepsilon}\left\|\int_L^\infty K_N(\xi)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}e^{-iux}d\xi\right\|_2^2.$$

$$\leqq\frac{1}{\varepsilon}\Big(\int_L^\infty|K_N|d\xi\Big)\Big(\int_L^\infty|K_N(\xi)|d\xi\int_{-\infty}^\infty|s(u+\varepsilon, x-\xi)-s(u-\varepsilon, x-\xi)|^2du\Big)$$

$$+\frac{1}{\varepsilon}\int_{-\infty}^\infty\left|\Big(\int_L^\infty K_N(\xi)e^{-iu\xi}d\xi\Big)\{s(u+\varepsilon, x)-s(u-\varepsilon, x)\}\right|^2du$$

$$\leqq 16\pi\sup|f|^2\Big(\int_L^\infty|K_N(\xi)|d\xi\Big)^2.$$

By the similar way we get

$$\frac{1}{\varepsilon}\left\|\int_{-\infty}^{-L}K_N(\xi)\Delta(u, x-\xi, \varepsilon)d\xi\right\|^2\leqq 16\,\pi\sup|f|^2\Big(\int_{-\infty}^{-L}|K_N(\xi)|^2d\xi\Big)^2.$$

If we take L sufficiently large and fix, then tend ε to 0. We can make the left hand side of (2.8) to be less than any positive number. Thus we have proved (2.8). Therefore we get

$$\text{(2.9)}\quad \overline{\lim_{\varepsilon\to0}}\sup_x\|t_N(u+\varepsilon,x)-t_N(u-\varepsilon,x)\|_2^2=\overline{\lim_{\varepsilon\to0}}\sup_x\|k_N(u)\{s(u+\varepsilon)-s(u-\varepsilon)\}\|_2^2.$$

Furthermore we have

$$\sup_{-\infty<u<\infty}|k(u)-k_N(u)|\leqq\|K-K_N\|_1\to0\quad(N\to\infty)$$

and

$$\frac{1}{\varepsilon}\|\{t(u+\varepsilon, x)-t(u-\varepsilon, x)\}-\{t_N(u-\varepsilon, x)-t_N(u-\varepsilon, x)\}\|_2^2$$

$$\leqq 16\sup_x|f|^2\|K-K_N\|_1\to0\quad(N\to0).$$

Therefore we obtain (2.7) from (2.9).

3. Spectrum of bounded function. Concerning to the definition of spectrum of functions, in particular as for bounded measurable functions, there are several ones. These are due to A. Beurling [2, 3, 6] and H. Pollard [10]. These are quite different apparently but are equivalent to each other. One of these is as follows:

Definition 1. Let f be a bounded measurable function on $(-\infty, \infty)$. By $\Lambda(f)$ we mean the set of real number λ with following property: if K is a function in $L_1(-\infty, \infty)$ such that

$$\text{(3.1)}\qquad (K*f)(x)=\int_{-\infty}^\infty K(x-y)f(y)dy=0$$

for all x in $(-\infty, \infty)$ the Fourier transform of K vanishes at $t=\lambda$.

Let us introduce the following two definitions of spectrum of f.

Definition 2. By $\Lambda_*(f)$ we mean the set of real number λ with the following property: if K is a function in $L_1(-\infty, \infty)$ such that

$$\text{(3.2)}\qquad (K*f)(x)=\int_{-\infty}^\infty K(x-y)f(y)dy\sim0$$

in the sense of (1.4), then the Fourier transform of K vanishes at $t=\lambda$.

Definition 3. By $\Lambda_{Wy}(f)$ we mean the set of real number λ with the following property: for any positive number $\delta>0$

$$\overline{\lim_{\varepsilon\to 0}} \sup_{-\infty<x<\infty} \frac{1}{\varepsilon}\int_{\lambda-\delta}^{\lambda+\delta} |s(u+\varepsilon, x)-s(u-\varepsilon, x)|^2 du>0. \tag{3.3}$$

Applying Theorems 1 and 3 we get immediately

Theorem 4. For any bounded measurable function we have

$$\Lambda_*(f)=\Lambda_{Wy}(f). \tag{3.4}$$

We omit details (cf. A. Beurling [4, 5] and H. Pollard [10, Theorem 4.1]).

Definition 4. By $\Lambda^*(f)$ we mean the set of real number λ with the following property: $e^{i\lambda x}$ is contained in the manifold which is spanned by dilatations of $f(t)$.

Then it is clearly

$$\Lambda^*(f)\supset\Lambda(f)\supset\Lambda_*(f). \tag{3.5}$$

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