# 37. *Tauberian Theorems Concerning the Summability Methods of Logarithmic Type*

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§ 1. In the recent papers the author proved some theorems concerning the summability methods of logarithmic type. (See [3, 4].) When a sequence $\{s_n\}$ is given we define the method $l$ as follows: If

$$t_0=s_0,\quad t_1=s_1,$$

(1) $$t_n=\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\quad(n\geq 2)$$

tends to a finite limit $s$ as $n\to\infty$, we say $\{s_n\}$ is summable $(l)$ to $s$ and write $\lim s_n=s(l)$. (See [2] p. 59, p. 87, [5] p. 32.)

On the other hand we define the method $L$ as follows: If

(2) $$f(x)=\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

tends to a finite limit $s$ as $x\to 1$ in the open interval $(0,1)$, we say that $\{s_n\}$ is summable $(L)$ to $s$ and write $\lim s_n=s(L)$. (See [1].)

When a series $\sum_{n=0}^{\infty}a_n$ is given we define the method $l$ and the method $L$ as before by putting

$$s_n=a_0+a_1+\cdots+a_n\quad(n\geq 0).$$

In the present note we shall prove the following two theorems.

**Theorem 1.** *If* $\sum_{n=0}^{\infty}a_n$ *is summable* $(l)$ *to* $s$, *and if*

(3) $$a_n=o\left(\frac{1}{n\log n}\right),$$

*then* $\sum_{n=0}^{\infty}a_n$ *converges to the same value.*

**Theorem 2.** *If* $\sum_{n=0}^{\infty}a_n$ *is summable* $(L)$ *to* $s$, *and if it satisfies* (3), *then* $\sum_{n=0}^{\infty}a_n$ *converges to the same value.*

Since the series summable $(l)$ is also summable $(L)$ to the same sum, Theorem 2 includes Theorem 1. (See [3].) However the proof of Theorem 1 seems to be fundamental, we shall prove Theorem 1 first.

§ 2. **Proof of Theorem 1.** From (1) we get

$$s_n-t_n=\frac{1}{\log n}\left\{s_n\log n-\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\right\}.$$

Since

$$\log n=1+\frac{1}{2}+\cdots+\frac{1}{n+1}+O(1)\quad\text{as}\quad n\to\infty,$$

we get

$$s_n - t_n = \frac{1}{\log n}\left\{s_n\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)+s_nO(1)-\right.$$
$$\left.-\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\right\}.$$

On the other hand we get, from (3),

$$(4)\qquad \frac{s_n}{\log n}=o(1) \quad \text{as} \quad n\to\infty.$$

Hence

$$s_n-t_n=\frac{1}{\log n}\left\{(s_n-s_0)+\frac{1}{2}(s_n-s_1)+\cdots+\frac{1}{n}(s_n-s_{n-1})\right\}+o(1)$$
$$=\frac{1}{\log n}\left\{(a_1+a_2+\cdots+a_n)+\frac{1}{2}(a_2+a_3+\cdots+a_n)+\right.$$
$$\left.+\cdots+\frac{1}{n}a_n\right\}+o(1)$$
$$=\frac{1}{\log n}\left\{\frac{2a_1}{2}+\frac{3a_2\left(1+\frac{1}{2}\right)}{3}+\frac{4a_3\left(1+\frac{1}{2}+\frac{1}{3}\right)}{4}+\right.$$
$$\left.+\cdots+\frac{(n+1)a_n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n+1}\right\}+o(1).$$

On the other hand we obtain, from (3),

$$\lim_{n\to\infty}(n+1)a_n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=0.$$

Since the method $l$ is regular, we can deduce

$$\lim_{n\to\infty}(s_n-t_n)=0,$$

obtaining $\lim_{n\to\infty} s_n = s$ from the assumption. (See [2] p. 59.)

This completes the proof of Theorem 1.

**Proof of Theorem 2.** From (2) we get, for $0<x<1$,

$$s_p-f(x)=\frac{-1}{\log(1-x)}\left\{\sum_{n=0}^{\infty}\frac{s_px^{n+1}}{n+1}-\sum_{n=0}^{\infty}\frac{s_nx^{n+1}}{n+1}\right\}$$
$$=\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{(s_p-s_n)}{n+1}x^{n+1}$$
$$=\frac{-1}{\log(1-x)}\sum_{n=0}^{p-1}\frac{(s_p-s_n)}{n+1}x^{n+1}+\frac{-1}{\log(1-x)}\sum_{n=p+1}^{\infty}\frac{(s_p-s_n)}{n+1}x^{n+1}$$

$=I+J$, say. Here we get

$$|I|\le\frac{-1}{\log(1-x)}\sum_{n=0}^{p-1}\frac{|s_p-s_n|}{n+1}.$$

If we put $x=1-\frac{1}{p}$, then

$$|I|\le\frac{1}{\log p}\sum_{n=0}^{p-1}\frac{|s_p-s_n|}{n+1}$$

$$\leq \frac{1}{\log p}\Big\{|a_1|+|a_2|+\cdots+|a_p|+$$

$$+\frac{1}{2}(|a_2|+|a_3|+\cdots+|a_p|)+\cdots+\frac{1}{p}|a_p|\Big\}$$

$$=\frac{1}{\log p}\left\{\frac{2|a_1|}{2}+\frac{3|a_2|\left(1+\frac{1}{2}\right)}{3}+\frac{4|a_3|\left(1+\frac{1}{2}+\frac{1}{3}\right)}{4}+\right.$$

$$\left.+\frac{(p+1)|a_p|\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)}{p+1}\right\}.$$

Since we get, from (3),

$$\lim_{p\to\infty}(p+1)|a_p|\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)=0,$$

we obtain

$$I=o(1) \quad \text{as} \quad p\to\infty,$$

from the regularity of the method $l$.

Next we get, for $0<x<1$,

$$|J|\leq \frac{-1}{\log(1-x)}\sum_{n=p+1}^{\infty}\frac{|s_n-s_p|}{n+1}x^{n+1}$$

$$\leq \frac{-1}{\log(1-x)}\sum_{n=p+1}^{\infty}\frac{|a_{p+1}|+|a_{p+2}|+\cdots+|a_n|}{n+1}x^{n+1}.$$

Being given any $\varepsilon>0$ we can find a $p_0=p_0(\varepsilon)$ such that

$$|a_p|<\frac{\varepsilon}{p\log p},$$

provided $p>p_0$. Hence, for $p>p_0$, we obtain

$$|a_{p+1}|<\frac{\varepsilon}{(p+1)\log(p+1)}<\frac{\varepsilon}{p+1}<\frac{\varepsilon}{p}$$

$$|a_{p+2}|<\frac{\varepsilon}{p+2}<\frac{\varepsilon}{p}$$

$$\cdots\cdots\cdots\cdot$$

$$|a_n|<\frac{\varepsilon}{n}<\frac{\varepsilon}{p},$$

so that, for $0<x<1$,

$$|J|\leq \frac{-1}{\log(1-x)}\sum_{n=p+1}^{\infty}\frac{\frac{\varepsilon}{p}(n-p)}{n+1}x^{n+1}$$

$$\leq \frac{-1}{\log(1-x)}\cdot\frac{\varepsilon}{p}\cdot\sum_{n=p+1}^{\infty}x^{n+1}$$

$$\leq \frac{-1}{\log(1-x)}\cdot\frac{\varepsilon}{p}\cdot\frac{1}{1-x}.$$

If we put $x=1-\frac{1}{p}$, then

$$|J|\leq \frac{1}{\log p}\cdot\frac{\varepsilon}{p}\cdot p=\frac{\varepsilon}{\log p}.$$

Hence we get $J=o(1)$ as $p\to\infty$. Consequently we have

$$\lim_{p\to\infty}\left\{s_p-f\left(1-\frac{1}{p}\right)\right\}=0,$$

obtaining $\lim\limits_{p\to\infty} s_p=s$ from the assumption of this theorem.

This completes the proof of Theorem 2.

## References

[1] D. Borwein: A logarithmic method of summability, J. London Math. Soc., **33**, 212–220 (1958).

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[5] O. Szász: Introduction to the Theory of Divergent Series, Cincinnati (1952).