35. On the Product of a Normal Space with a Metric Space

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(Comm. by Kinjirô KUNUGI, M.J.A., March 12, 1963)

Let X be a topological space. Then the topological product of X with every metrizable space is proved to be normal for the following three cases.

I. X is paracompact and perfectly normal (E. Michael [2]).

II. X is paracompact and topologically complete in the sense of E. Čech (Z. Frolik [1]).

III. X is countably compact and normal (A. H. Stone [4]).

Quite recently E. Michael [3] has shown that the product space $X \times Y$ is not normal in general even if X is a hereditarily paracompact Hausdorff space with the Lindelöf property and Y is a separable metric space.

In view of these facts it is desirable to find a necessary and sufficient condition for X to possess the property that the product space $X \times Y$ be normal for any metrizable space Y. This problem, however, was open until now (cf. H. Tamano [5]). The purpose of this note is to give a solution to this problem. The proofs and the details of the results will be published elsewhere.

1. Let us consider the following condition for a topological space X.

For any set Ω of indices and for any family $\{G(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega;\ i=1,2,\cdots\}$ of *open* subsets of X satisfying the condition

(1) $G(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_{i+1} \in \Omega$ and for $i=1,2,\cdots$

there exists a family $\{F(\alpha_1,\cdots,\alpha_i) \mid \alpha_1,\cdots,\alpha_i \in \Omega,\ i=1,2,\cdots\}$ of *closed* subsets of X satisfying the following two conditions:

(2) $F(\alpha_1,\cdots,\alpha_i) \subset G(\alpha_1,\cdots,\alpha_i)$ for $\alpha_1,\cdots,\alpha_i \in \Omega$.

(3) If $\bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i) = X$, then $\bigcup_{i=1}^{\infty} F(\alpha_1,\cdots,\alpha_i) = X$.

We shall say that X is a *P-space* if X satisfies the above condition.

As is well known, a normal space X is countably paracompact if and only if for any countable open covering $\{G_i\}$ of X with $G_i \subset G_{i+1}$, $i=1,2,\cdots$ there exists a countable closed covering $\{F_i\}$ of X such that $F_i \subset G_i$, $i=1,2,\cdots$. Hence a normal P-space is always countably paracompact. On the other hand, it follows from an example of Michael [3], in view of our Theorem 2.1 below, that a hereditarily paracompact Hausdorff space with the Lindelöf property is not necessarily a P-space.

Theorem 1.1. *Countably compact spaces and perfectly normal spaces are P-spaces.*

Normal P-spaces have many properties analogous to those of countably paracompact normal spaces, but these properties will not be stated here.

2. Our main theorems read as follows.

Theorem 2.1. *Let X be a topological space. In order that the product space $X\times Y$ be normal for any metrizable space Y it is necessary and sufficient that X be a normal P-space.*

Theorem 2.2. *Let X be a topological space. Then the product space $X\times Y$ is normal for any separable metric space Y if and only if X is a normal space such that for any family $\{G(\varepsilon_1,\cdots,\varepsilon_i)\mid \varepsilon_1,\cdots,\varepsilon_i=0,1;\ i=1,2,\cdots\}$ of open sets of X with $G(\varepsilon_1\cdots,\varepsilon_i)\subset G(\varepsilon_1,\cdots,\varepsilon_i,\varepsilon_{i+1})$, $i=1,2,\cdots$ there exists a family $\{F(\varepsilon_1,\cdots,\varepsilon_i)\mid \varepsilon_1,\cdots,\varepsilon_i=0,1;\ i=1,2,\cdots\}$ of closed sets of X satisfying the two conditions below:*

(2)′ $$F(\varepsilon_1,\cdots,\varepsilon_i)\subset G(\varepsilon_1\cdots,\varepsilon_i);$$

(3)′ $$\text{if } \bigcup_{i=1}^{\infty} G(\varepsilon_1,\cdots,\varepsilon_i)=X, \text{ then } \bigcup_{i=1}^{\infty} F(\varepsilon_1,\cdots,\varepsilon_i)=X.$$

3. There are no intimate relations between paracompact spaces and P-spaces. However, we can prove the following theorems.

Theorem 3.1. *Let X be a normal P-space and Y a metrizable space. If X is paracompact, then the product space $X\times Y$ is paracompact.*

Theorem 3.2. *If X is a normal space which satisfies the condition of Theorem 2.2 and has the Lindelöf property, and if Y is a separable metric space, then the product space $X\times Y$ is a normal space with the Lindelöf property.*

4. The following theorem is a generalization of a theorem of Michael [2].

Theorem 4.1. *If X is a perfectly normal space and Y is a metrizable space, then the product $X\times Y$ is perfectly normal.*

Proof. The normality of $X\times Y$ follows immediately from Theorems 1.1 and 2.1. Since it is proved by Michael [2] that any open subset of $X\times Y$ is an F_σ-set, we have at once Theorem 4.1. However, it is possible to give a direct proof which does not appeal to Theorems 1.1 and 2.1. The following is such a proof.

Let $\{V_{i\alpha}\mid \alpha\in\Omega_i,\ i=1,2,\cdots\}$ be an open basis of Y such that $\{V_{i\alpha}\mid\alpha\in\Omega_i\}$ is locally finite for each i. Let H be any open set of $X\times Y$. Then there exist open sets $G_{i\alpha}$ of X such that $H=\bigcup\{G_{i\alpha}\times V_{i\alpha}\mid \alpha\in\Omega_i,\ i=1,2,\cdots\}$. By the perfect normality of X and Y there exist, for each i and $\alpha\in\Omega_i$, continuous maps

$$\varphi_{i\alpha}\colon X\to I,\quad \psi_{i\alpha}\colon Y\to I,\quad (I=[0,1])$$

such that

$$G_{i\alpha}=\{x\mid\varphi_{i\alpha}(x)>0\},\ V_{i\alpha}=\{y\mid\psi_{i\alpha}(y)>0\}.$$

If we put

$$h_i(x,y)=\sum_{\alpha\in\Omega_i}\varphi_{i\alpha}(x)\psi_{i\alpha}(y),\quad\text{for}\quad x\in X,\ y\in Y,$$

then h_i is continuous over $X\times Y$ for each i since $\{G_{i\alpha}\times V_{i\alpha}\mid\alpha\in\Omega_i\}$ is locally finite for each i. If we put further

$$h(x,y)=\sum_{i=1}^{\infty}\frac{h_i(x,y)}{2^i(1+h_i(x,y))},$$

then h is continuous over $X\times Y$ and we have $H=\{(x,y)\mid h(x,y)>0\}$. This proves the perfect normality of $X\times Y$.

5. We shall say that a topological space X is an *M-space* if there exists a normal sequence $\{\mathfrak{U}_i\mid i=1,2,\cdots\}$ of open coverings of X such that if a family $\mathfrak{K}$ consisting of a countable number of subsets of X has the finite intersection property and contains a subset of $\mathrm{St}(x_0,\mathfrak{U}_i)$ for each i and for some fixed point x_0 of X, then $\bigcap\{\overline{K}\mid K\in\mathfrak{K}\}\neq\phi$.

Theorem 5.1. *An M-space is a P-space but not conversely.*

Theorem 5.2. *Any paracompact Hausdorff space which is topologically complete in the sense of E. Čech is an M-space.*

Theorem 5.3. *The following two statements are equivalent for X.*

I. *X is an M-space.*

II. *There exists a closed continuous map φ of X onto a metrizable space S such that $\varphi^{-1}(s)$ is countably compact for each point s of S.*

Theorem 5.4. *The topological product of a paracompact Hausdorff P-space with a paracompact Hausdorff M-space is a paracompact P-space.*

References

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