45. *Continuity of Path Functions of Strictly Stationary Linear Processes*

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Let $X(t)$, $-\infty<t<+\infty$, be a mean continuous purely non-deterministic weakly stationary process with $EX(t)=0$. Then, by Karhunen [5], $X(t)$ can be expressed in the following form.

$$(1) \qquad X(t)=\int_{-\infty}^{t} g(t-u)\,dZ(u),$$

where the function g is in $L_2(R)$ and dZ is an orthogonal random measure such that $E(dZ(u))^2=du$. Further, let $\mathfrak{M}_t(X)$, $\mathfrak{M}(X)$ and $\mathfrak{M}_t(Z)$ be closed linear manifolds spanned by $\{X(\tau);\ \tau\leq t\}$, $\{X(\tau);\ -\infty<\tau<+\infty\}$ and $\{Z(\tau)-Z(\tau');\ \tau,\tau'\leq t\}$, respectively. We can take g and dZ to satisfy $\mathfrak{M}_t(X)=\mathfrak{M}_t(Z)$, uniquely up to the constant multiple with absolute value one.

Next, following P. Lévy and Hida-Ikeda [2], we call $X(t)$ a *linear process* if $\mathfrak{M}_t(X)$ and $\mathfrak{M}_t^{\perp}(X)=\{$the orthogonal complement of $\mathfrak{M}_t(X)$ in $\mathfrak{M}(X)\}$ are mutually independent for each t.

PROPOSITION. *Let $X(t)$ be a strictly stationary process with canonical representation of the form* (1). *Then $X(t)$ is a linear process if and only if $Z_a(t)=Z(t)-Z(a)$, $t\geq a$, is a temporally homogeneous additive process for each a.*

The proof of 'if' part is found in Hida-Ikeda [2]. 'Only if' part is easily proved by the definition of canonical representation.

In the following we assume $X(t)$ to be strictly stationary and linear. We want to investigate properties of its path functions.

An additive process which is continuous in probability may be considered as a Lévy process by taking an appropriate version. Hence, by Lévy-Itô's decomposition, we can write

$$(2) \qquad Z(t)-Z(a)=\sqrt{v}(B_0(t)-B_0(a))+P(t)-P(a),$$

where $B_0(t)$ is the standard Brownian motion and $P(t)-P(a)$ is the Poisson part. Then (1) and (2) imply

$$(3) \qquad X(t)=\sqrt{v}\int_{-\infty}^{t} g(t-u)\,dB_0(u)+\int_{-\infty}^{t} g(t-u)\,dP(u).$$

We denote the first term on the right side by $X_1(t)$ and the second by $X_2(t)$. $X_1(t)$ is a Gaussian stationary process and the properties of its path functions are investigated by Hunt [3] and Belayev [1]. So we shall treat $X_2(t)$ and give a sufficient condition for the continuity of its path functions.

Suppose that $X_1(t)=0$, that is

$$X(t)=\int_{-\infty}^{t} g(t-u)\,dP(u). \tag{4}$$

We use the stochastic integral of Itô [4] §9. Then using Itô's notation, $P(t)$ is expressed as follows,

$$P(t,\omega)-P(a,\omega)=\int_{a}^{t}\int_{|s|>0} f(s)q(duds,\omega), \tag{5}$$

where $\int_{|s|>0} f(s)^2\frac{ds}{s^2}<+\infty$ since $Z_a(t)$ has a finite variance. A version of $X(t)$ is written by the stochastic integral: thus we represent $X(t)$ as follows,

$$X(t)=\int_{-\infty}^{t}\int_{|s|>0} g(t-u)f(s)q(duds). \tag{6}$$

THEOREM. *If g and f in the expression* (6) *satisfy the following conditions, $X(t)$ has a version of which almost all path functions are continuous.*

(7) *$g(\tau)$ is continuous in $\tau\in[0,\infty)$ and $g(0)=0$.*

$$\int_{|s|>0} |f(s)|\frac{ds}{s^2}<+\infty. \tag{8}$$

(9) *There exist $N,\delta>0$ and $g_0\in L_1(N,\infty)$ such that for any $t>N$, $\sup_{t<\tau<t+\delta}|g(\tau)|\le g_0(t)$.*

PROOF. It suffices to prove that for every interval $[a,a+\delta]$ of length δ,

$$P(\lim_{h\to 0}\sup_{\substack{r,r'\in[a,a+\delta]\\ 0<|r'-r|<h}}|X(r')-X(r)|=0)=1 \tag{10}$$

where r and r' run over rational numbers.

Suppose $r'>r$. Then

$$\begin{aligned}
&\sup_{\substack{r,r'\in[a,a+\delta]\\ 0<r'-r<h}}|X(r',\omega)-X(r,\omega)|\\
&\le \sup_{\substack{r,r'\in[a,a+\delta]\\ 0<r'-r<h}}\left|\int_{r}^{r'}\int_{|s|>0} g(r'-\tau)f(s)q(d\tau ds,\omega)\right|\\
&+\sup_{\substack{r,r'\in[a,a+\delta]\\ 0<r'-r<h}}\left|\int_{-N}^{r}\int_{|s|>0}(g(r'-\tau)-g(r-\tau))f(s)q(d\tau ds,\omega)\right|\\
&+\sup_{r\in[a,a+\delta]}2\left|\int_{-\infty}^{-N}\int_{|s|>0} g(r-\tau)f(s)q(d\tau ds,\omega)\right|.
\end{aligned}$$

We denote the terms on the right side by $I_1(h,\omega)$, $I_2(h,N,\omega)$ and $I_3(N,\omega)$, respectively.

i) Part I_1: By the condition (7), for each $\varepsilon>0$, there exists $h>0$ such that $|g(u)|<\varepsilon$, for $0<u<h$. Then, from Itô's notation $q=p-\frac{d\tau ds}{s^2}$,

$$\sup_{\substack{r'-r<h \\ r, r' \in [a, a+\delta]}} \left| \int_r^{r'} \int_{|s|>0} g(r'-\tau) f(s) q(d\tau ds, \omega) \right|$$

$$\leq \varepsilon \lim_{n\to\infty} \int_a^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| p(d\tau ds, \omega) + \varepsilon \lim_{n\to\infty} \int_a^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \frac{d\tau ds}{s^2}.$$

The limit of the first term on the right is finite for almost all ω, since

$$E \lim_{n\to\infty} \int_a^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| p(d\tau ds, \omega) = \lim_{n\to\infty} \int_a^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \frac{d\tau ds}{s^2}.$$

The second limit is also finite. Then we have

$$P(\lim_{h\to 0} I_1(h, \omega) = 0) = 1.$$

ii) Part I_3: We prove that there exists, for each $\varepsilon>0$, a sufficiently large $N(\varepsilon, \omega)$ for which we have $I_3 \leq \varepsilon$. From the condition (9), if N is sufficiently large,

$$\sup_{r \in [a, a+\delta]} \left| \int_{-\infty}^{-N} \int_{|s|>0} g(r-\tau) f(s) q(d\tau ds, \omega) \right|$$

$$\leq \lim_{n\to\infty} \int_{-\infty}^{-N} \int_{|s|>\frac{1}{n}} |g_0(a-\tau) f(s)| p(d\tau ds, \omega) + \lim_{n\to\infty} \int_{-\infty}^{-N} \int_{|s|>\frac{1}{n}} |g_0(a-\tau) f(s)| \frac{d\tau ds}{s^2}.$$

For the first term on the right, we have

$$E \lim_{n\to\infty} \int_{-\infty}^{-N} \int_{|s|>\frac{1}{n}} |g_0(a-\tau) f(s)| p(d\tau ds, \omega)$$

$$= \int_{-\infty}^{-N} |g_0(a-\tau)| d\tau \cdot \int_{|s|>0} |f(s)| \frac{ds}{s^2} \to 0 \quad (N \to \infty),$$

by the conditions (8) and (9). The same is true for the second term on the right. That is, I_3 tends to zero in mean. Then we can take a subsequence N' such that

$$P(\lim_{N'\to\infty} I_3(N') = 0) = 1$$

iii) Part I_2: Let $M(\omega)$ be given by

$$M(\omega) = \lim_{n\to\infty} \int_{-N'}^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \left(p(d\tau ds, \omega) + \frac{d\tau ds}{s^2} \right).$$

Then $M(\omega)$ is finite for almost all ω. Because, for fixed N',

$$E \lim_{n\to\infty} \int_{-N'}^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \left(p(d\tau ds, \omega) + \frac{d\tau ds}{s^2} \right)$$

$$\leq \lim_{n\to\infty} 2 \int_{-N'}^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \frac{d\tau ds}{s^2} < +\infty.$$

From the condition (7), there exists, for $N'(\varepsilon, \omega)$ of ii) and each $\varepsilon>0$, an $h(\omega)$ such that

$$\sup_{\substack{r'-r<h \\ r,r'\in[a,a+\delta] \\ \tau\in[-N',r]}} |g(r'-\tau)-g(r-\tau)| < \frac{\varepsilon}{M(\omega)}.$$

Then we have

$$I_2 \leq \frac{\varepsilon}{M(\omega)} \lim_{n\to\infty} \int_{-N'}^{a+\delta} \int_{|s|>\frac{1}{n}} |f(s)| \left(p(d\tau ds, \omega) + \frac{d\tau ds}{s^2} \right) = \varepsilon.$$

Combining i), ii) and iii), we obtain (10), so that the proof of our theorem is complete.

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