67. *Semigroups of Positive Integer Vectors*[1)]

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1. Consider the set I of all the vectors $(x_1,\cdots,x_i,\cdots,0,\cdots)$ with countable components of non-negative integers where a finite number of x_i's are positive and the remaining are 0, but all are not 0. The addition is defined as follows:

$$(x_1,\cdots,x_i,\cdots)+(y_1,\cdots,y_i,\cdots)=(x_1+y_1,\cdots,x_i+y_i,\cdots)$$

in which x_i+y_i is the usual addition of integers. Then I forms a semigroup with respect to the addition. We want to determine all the subsemigroups of I from the standpoint of the bases, and by using these results, we can determine the subsemigroups of the multiplicative semigroup of some positive integer vectors. The detailed proof will be given in another paper.

When n is a positive integer and $a\in I$, na denotes $\underbrace{a+\cdots+a}_{n}\in I$. For convenience, I is considered as a subsemigroup of the module R of all vectors whose components are rational numbers and all are 0 except a finite number of x_i's where scalar-multiplying is regarded as an operator, that is, if λ is a rational number and $a\in I$, $\lambda a =\lambda(a_1,\cdots,a_i,\cdots)=(\lambda a_1,\cdots,\lambda a_i,\cdots)$ where λa_i is the usual product of λ and a_i.

2. Let M be a subsemigroup of I. If M can be embedded into the subsemigroup $I_k=\{x\in I\mid x=(x_1,\cdots,x_k,0\cdots),\ x_i=0,\ i>k\}$ and never into $I_{k'}(k'<k)$, then the dimension of M is said to be k and denoted by $\dim. M=k$ or M is called a k-dimensional subsemigroup of I, or, simply, k-dimensional semigroup. If there is no finite k, the dimension of M is said to be infinite. If M is k-dimensional, every element x of M can be expressed as $x=(x_1,\cdots,x_k)$ without loss of generality. A subset B of a finite or infinite dimensional M is called a generator system if each element a of M is of the form $a=\lambda_1 b_1+\cdots+\lambda_m b_m$ where λ_i are positive integers and $b_i\in B$ and m is not fixed. If B is minimal generator system of M in the sense of the inclusion relation, then B is called a basis of M. If B consists of finite elements, M is said to have a finite basis, and the number of elements of B is called the basis-order of M.

Lemma 1. *A subsemigroup M of I has a unique basis.*

A finite number of elements $b_1,\cdots,b_m$ in M are said to be linearly

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independent if $\xi_1 b_1 + \cdots + \xi_m b_m = 0$ (where 0 is the zero-vector and ξ_i's are rational numbers) implies $\xi_1 = \cdots = \xi_m = 0$. If M is n-dimensional, the basis B contains a subset B_L composed of n elements which are linearly independent. B_L is called a linearly independent basis of M. B_L is not unique in general.

Let M be an n-dimensional semigroup ($n < \infty$). Every element of the basis B of M can be uniquely expressed by a linear combination of elements of B_L: $\xi_1 b_1 + \cdots + \xi_n b_n$ where ξ_i's are rational numbers and $b_i \in B_L$.

Let $\boldsymbol{B}$ be the matrix of (m, n)-type, $n < \infty$, all the row-vectors of which form a basis of the n-dimensional M. We note that m denotes an ordinal number $\leqq \omega$, ω being the first non-finite ordinal; and if the cardinal of m is denoted by $\overline{m}$, $\overline{m}$ is the basis-order of M, $\overline{m} \leqq \aleph_0$. Let $\boldsymbol{B}_L$ be the matrix of (n, n)-type all the row-vectors of which form a linearly independent basis of M. is called the basis matrix of M and $_L$ is called the linearly independent basis matrix of M. Then we can determine a matrix of (m, n)-type $\boldsymbol{P} = (P_{ij})$ where P_{ij} is a rational number such that $= \boldsymbol{P}\boldsymbol{B}_L$. $\boldsymbol{P}$ is called the basis transformation matrix of M and we may denote $\boldsymbol{P}(M)$ if M is specified. We define the equivalence of the matrices of (m, n)-type: Two matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be equivalent if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are same type and if there are a permutation matrix $\boldsymbol{U}$ of (m, m)-type and a regular matrix $\boldsymbol{V}$ of (n, n)-type such that $\boldsymbol{UPV} = \boldsymbol{Q}$. By a permutation matrix we mean a matrix in which every row and every column contain only one non-zero element 1.

Theorem 1. *Let M and M' be n- and n'-dimensional semigroups ($n < \infty$, $n' < \infty$). M and M' are isomorphic iff $n = n'$ and $\boldsymbol{P}(M)$ and $\boldsymbol{P}(M')$ are equivalent.*

Corollary. *Suppose dim. M = dim. $M' = 1$ and let $\{b_1, \cdots, b_k\}$ and $\{b'_1, \cdots, b'_{k'}\}$ be the bases of M and M' respectively such that*

$$b_i < b_j, \quad b'_i < b'_j \text{ for } i < j.$$

M and M' are isomorphic iff $k = k'$ and $b_1/b'_1 = \cdots = b_k/b'_k$ (= rational number).

Let $n \geq 2$. Consider a mapping φ of an n-dimensional M into the $(n-1)$-dimensional Euclidean space E^{n-1} as follows:

$$\varphi(a) = \varphi((a_1, \cdots, a_n)) = (a_2/|a|, \cdots, a_n/|a|)$$

where

$$|a| = \sum_{i=1}^{n} a_i.$$

$\varphi(a)$ is called the ratio image of a, and $\varphi(M)$ is called the ratio image of M. If $a = \lambda_1 b_1 + \cdots + \lambda_n b_n$, $b_i \in B_L$, λ_i rational numbers not all zero, then

$$\varphi(a) = \mu_1 \varphi(b_1) + \cdots + \mu_n \varphi(b_n), \ \sum_{i=1}^{n} \mu_i = 1$$

where

$$\mu_i = \frac{m_i}{m_1 + \cdots + m_n}, \quad m_i = \lambda_i |b_i|.$$

If M is 1-dimensional the basis-order of M is finite; it is equal to at most the element of M in the sense of the usual ordering. This result was proved by K. Tetsuya and one of the authors [1]. However, if M is n-dimensional ($n>1$), it is not true in general.

Theorem 2. *An n-dimensional semigroup M has a finite basis iff the ratio image $\varphi(M)$ of M is contained in a convex polyhedron generated by $\varphi(B'_L)$ on rational numbers where B'_L is a subset of B_L.*

3. Suppose that there is given an infinite sequence of d_i-dimensional semigroups S_i, $d_i<\infty$, $d_i<d_j$, $i<j$, with the system of isomorphisms f_i^j of S_i into $S_j (i<j)$ satisfying $f_j^k f_i^j(x) = f_i^k(x)$, $x \in S_i$. The sequence $\{S_i\}$ with isomorphisms $\{f_i^j\}$ is denoted by $\{S_i; f_i^j\}$. We can define the limit semigroup of a sequence $\{S_i; f_i^j\}$ just as the limit group of a sequence of groups. For the proof of Theorems 4, 5 it is necessary to keep $\{s_i; f_i^j\}$ in certain standard form, but the explanation is omitted here.

Theorem 3. *Let $\{S_i; f_i^j\}$ be a sequence of finite dimensional semigroups. Then the limit semigroup S of $\{S_i; f_i^j\}$ is an infinite dimensional semigroup. Conversely an infinite dimensional semigroup S is the limit semigroup of a certain sequence $\{S_i; f_i^j\}$ of finite dimensional subsemigroups S_i of S.*

Theorem 4. *Let S and T be the limit semigroups of $\{S_i; f_i^j\}$ and $\{T_p; g_p^q\}$ respectively. S and T are isomorphic iff any S_i is isomorphic into certain $T_{i'}$, under the mapping $\varphi_i^{i'}$ and any T_p is isomorphic into certain $S_{p'}$ under $\psi_p^{p'}$ such that $f_i^j(x) = \psi_{i'}^j \varphi_i^{i'}(x)$ and $g_p^q(y) = \varphi_{p'}^q \psi_p^{p'}(y)$, $x \in S_i$, $y \in T_p$.*

Let $B^{(i)}$ and $B_L^{(i)}$ be the basis and the linearly independent basis of n_i-dimensional S_i $(i=1, 2, \cdots)$ respectively. We may assume $f_{i-1}^i (B_L^{(i-1)}) \subset B_L^{(i)}$. Then the basis B and a linearly independent basis B_L of the limit S of $\{S_i; f_i^j\}$ are given as follows;

$$B = \bigcup_{i=1}^{\infty} B^{(i)}, \qquad B_L = \bigcup_{i=1}^{\infty} B_L^{(i)}.$$

Since we see $f_i^{i+1}(B^{(i)}) \subset B^{(i+1)}$ and $f_i^{i+1}(B^{(i)} - B_L^{(i)}) \subset B^{(i+1)} - B_L^{(i+1)}$, we have one of the basis transformation matrix $\boldsymbol{P}(S_{i+1})$ of S_{i+1} as follows:

$$\boldsymbol{P}(S_1) = \left[\begin{array}{c} E^{(1)} \\ \hline \Lambda_1 \end{array}\right], \quad \boldsymbol{P}(S_{i+1}) = \left[\begin{array}{c|c} \boldsymbol{P}(S_i) & 0 \\ \hline 0 & E^{(i+1)} \\ \hline \multicolumn{2}{c}{\Lambda_{i+1}} \end{array}\right], \quad i=1, 2, \cdots$$

where, if $\boldsymbol{P}(S_i)$ is of (m_i, n_i)-type, then $\boldsymbol{P}(S_{i+1})$ is of (m_{i+1}, n_{i+1})-type, $m_{i+1} \leqq m_i + \omega$; and $E^{(i+1)}$ is the identity matrix of degree $n_{i+1} - n_i$.

Therefore, we can define the basis transformation matrix $P(S)$ as as follows:

$$\boldsymbol{P}(S)=\left[\begin{array}{cc:c:c}
\boldsymbol{P}(S_i) & 0 & & \\
\hdashline
0 & E^{(i+1)} & 0 & \\
\hdashline
\multicolumn{2}{c:}{\Lambda_{i+1}} & & \\
\hdashline
\multicolumn{2}{c:}{0} & E^{(i+2)} & 0 \\
\hdashline
\multicolumn{3}{c:}{\Lambda_{i+2}} & \\
\hdashline
 & & & \ddots
\end{array}\right]$$

where $\boldsymbol{P}(S)$ is of type (m, n)

$$m \leqq \omega+\omega+\cdots=\omega^2,\ n \leqq \omega.$$

The matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ of type (m, n) are said to be equivalent if there are a permutation matrix $\boldsymbol{U}$ of type-(m, n) and a regular matrix $\boldsymbol{V}$ of type-(n, n) such that $\boldsymbol{UPV}=\boldsymbol{Q}$. We note that an (ω, ω)-matrix $\boldsymbol{V}=(v_{ij})$, $i, j \geqq 1$, is called regular if $\boldsymbol{V}$ contains regular submatrices, $\boldsymbol{V}_{k_s}=(v_{ij})$, $1 \leqq i \leqq k_s$, $1 \leqq j \leqq k_s$, $s=1, 2, \cdots$.

Then we have

Theorem 5. *Let S and T be infinite dimensional semigroups. S and T are isomorphic iff $\boldsymbol{P}(S)$ is equivalent to $\boldsymbol{P}(T)$.*

Thus we see that, for a finite or infinite dimensional semigroup S, the basis transformation (m, n)-type matrix $\boldsymbol{P}(S)$ over rational numbers is uniquely determined within equivalence. $\boldsymbol{P}(S)$ is characterized by the following conditions. Let $\boldsymbol{P}(S)=(p_{ij})$.

(1) Any row vector of $\boldsymbol{P}$ cannot be generated by the other row vectors of $\boldsymbol{P}$.

(2) Any row vector a of $\boldsymbol{P}$ is non-negative where $a=(a_1, \cdots, a_i, \cdots)$ is called negative if $a_i \leqq 0$ for all i.

(3) There is a positive rational column vector

$$x=\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix},\quad x_i \geqq 0$$

such that $a \cdot x = a_1 x_1 + \cdots + a_n x_n > 0$ for all row vectors a of $\boldsymbol{P}$.

(4) There is a positive integer l such that all $l p_{ij}$ are integers.

In particular, if m and n are finite, (3) and (4) are redundant. Also we remark that the additive semigroup S is characterized by a commutative free semigroup with generating relations which are expressed by using the matrix $\boldsymbol{P}$.

4. Consider the set J of all the vectors $(x_1, \cdots, x_i, 1, \cdots)$ with countable components of positive integers where a finite number of x_i's are >1 and the remaining are 1, but all are not 1. The multiplication is defined as follows:

$$(x_1,\cdots,x_i,\cdots)(y_1,\cdots,y_i,\cdots)=(x_1y_1,\cdots,x_iy_i,\cdots)$$

in which x_iy_i is the usual multiplication of integers. In this section, a subsemigroup N of J shall be called a multiplicative semigroup (in J). To avoid confusion, we shall call a semigroup in the previous sections an additive semigroup (in I). We can define the dimension, the basis, the basis-order of a multiplicative semigroup N in the same way as those of the additive semigroup. The dimension of N which we shall denote by m-dim. $N=\lambda$. We can easily prove that Lemma 1 holds even in this case.

Let $a=(a_1,\cdots,a_i,\cdots)$ be any element of N. a_i has the expression of the product of prime factors:

$$a_i=p_{i1}^{\alpha_{i1}}p_{i2}^{\alpha_{i2}}\cdots p_{is}^{\alpha_{is}}.$$

For a fixed i, μ_i denotes the cardinal of the set of the distinct primes which appear in the factorization of a_i, a running throughout N and let $\mu=\max.\mu_i$ which we call the primary order of N.

Let N be a multiplicative semigroup with the m-dimension λ, the basis-order μ, the primary order ν.

Theorem 6. (1) *If $\lambda<\infty$ and $\nu<\infty$, then N is isomorphic to an additive semigroup which has finite dimension and of the same basis-order as N.*

(2) *If $\lambda=\infty$ or $\nu=\infty$, then $\mu=\infty$ and N is isomorphic to an additive semigroup which has infinite dimension and infinite basis-order.*

5. Again consider the set I in which the multiplication is defined in the same way as in J. Then the semigroup is denoted by $I^{\times}$. Let K be a subsemigroup of $I^{\times}$ and a be any element of K: $a=(a_1,\cdots,a_i,\cdots)$. $D(a)$ is defined to be the set of indices i such that $a_i\neq 0$.

We see

$$D(a)\neq\square \quad \text{for every } a$$
$$D(a) \text{ is a finite subset of } \{1,2,\cdots\}$$
$$D(ab)=D(a)\cap D(b).$$

The system $\{D(a);a\in K\}$ is a semilattice with respect to $\cap$, and hence we get a semilattice decomposition of K under the homomorphism $a\to D(a)$. Cleary the inverse image of $D(a)$ is a multiplicative semigroup of finite dimension.

Theorem 7. *A subsemigroup K of $I^{\times}$ is isomorphic to a semilattice of finite dimensional multiplicative semigroups in J.*

Reference

[1] T. Tamura and K. Tetsuya: On base of infinite quasi-power semigroup, Shikoku Sugaku Sijo Danwa, 295–298 (1953). This is written in Japanese and it has not been published in English.