66. A Note on Rings of which any One-sided Quotient Rings are Two-sided

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0. A ring S is called a J-ring if the left and right singular ideals vanish. For any J-ring S we may construct the maximal left quotient ring $\bar{S}_1$ and the maximal right quotient ring $\bar{S}_r$.

A left ideal A of a ring S is called closed if there exists a left ideal B such that A is maximal among left ideals disjoint to B. If S is a J-ring it is known that the set of closed left ideals forms a complete complemented modular lattice $L(S)$. Similarly we define closed right ideals, and denote the lattice of closed right ideals by $R(S)$.

We shall show the following two theorems:

Theorem 1. Let S be a J-ring, and suppose that both $L(S)$ and $R(S)$ are atomic. Then the following conditions are equivalent:

(A_1) The right annihilator of an atom of $L(S)$ is a dual atom of $R(S)$.

(B_1) For any atom A of $L(S)$ there exists an atom B of $R(S)$ such that $A\cap B\neq 0$.

(C_1) The right annihilator of the sum of atoms of $R(S)$ is zero.

Theorem 2. Let S be a J-ring. Suppose that both $L(S)$ and $R(S)$ are finite dimensional. Then $\bar{S}_1=\bar{S}_r$ if and only if S satisfies (A_1) and its right-left symmetry (A_r).

Similar results have been obtained by R. E. Johnson too in a completely different way.

1. We denote by $1(*)$ ($r(*)$ resp.) the left (right resp.) annihilator of $*$.

Proof of Theorem 1. $(A_1)\Rightarrow(B_1)$. Let X be an atom of $L(S)$, and let $0\neq x\in X$. Then $r(X)$ is a dual atom of $R(S)$ by assumption, and so $r(x)=r(X)$ since any annihilator right ideal is closed. Let A and B be nonzero right ideals contained in xS. Then we may suppose that $A=xA'$ and $B=xB'$ for some right ideals A' and B' which contain $r(x)$ properly. Now A' and B' are large, and so is $A'\cap B'$. Hence $0\neq x(A'\cap B')\subset A\cap B$. This shows that xS is uniform, therefore the closure of xS is an atom of $R(S)$. Since the closure contains x, this proves (B_1).

$(B_1)\Rightarrow(C_1)$. Let P be the sum of atoms of $R(S)$. Then $P\cap X$

$\neq 0$ for any atom X of $L(S)$ by assumption. Hence $1(r(P)) \cap X \neq 0$. Since $1(r(P))$ is closed, it follows from this that $1(r(P))=S$, and hence $r(P)=0$.

$(C_1) \Rightarrow (A_1)$. Let X be an atom of $L(S)$, and P the sum of atoms of $R(S)$. Then $PX \neq 0$ by assumption. Hence $YX \neq 0$ for some atom Y of $R(S)$, therefore $X \cap Y \neq 0$. Let $0 \neq x \in X \cap Y$. Since Y is an atom of $R(S)$, xS is uniform, and also is $S/r(x)$. It follows from this that $r(x)$ is a dual atom of $R(S)$. Now $1(r(x))$ is closed, and contains a nonzero elment x in common with X. Hence $1(r(x)) \supset X$, and so $r(x) = r(X)$. Thus, $r(X)$ is a dual atom of $R(S)$, completing the proof.

2. **Theorem 3.** Let S be a J-ring. Suppose that $\dim L(S) = \dim R(S) < \infty$. Then the condition (A_1) is equivalent to the following:

(K_1) If $A \cap B = 0$, $B \neq 0$ for left ideals A and B of S, then $r(A) \neq 0$.

Proof. $(A_1) \Rightarrow (K_1)$. Let $A \cap B = 0$, $B \neq 0$ for left ideals A, B. Then the closure C of A is not equal to S, and hence it is a join of k atoms of $L(S)$, k being smaller than the dimension of $L(S)$. If $r(C)=0$, then the intersection of the right annihilators of these atoms of $L(S)$ is zero. In virtue of (A_1) it follows from this that $\dim R(S) \leq k$, a contradiction. Thus, $r(C) \neq 0$, and $r(A) \neq 0$.

$(K_1) \Rightarrow (A_1)$. We suppose that S satisfies (K_1). Hence by ([1]; Theorem 2.2) S fulfills (K_1'); that is, every closed left ideal is annihilator. Let X be an atom of $L(S)$. $L(S)$ is now the lattice of annihilator left ideals, and hence it is dually isomorphic to the lattice $R'(S)$ of annihilator right ideals. Thus, $r(X)$ is a dual atom of $R'(S)$. Since $R'(S)$ is contained in $R(S)$, and since $\dim R'(S) = \dim L(S) = \dim R(S)$, it is easily seen that $r(X)$ is a dual atom of $R(S)$, completing the proof.

Proof of Theorem 1. Suppose that $\bar{S}_1 = \bar{S}_r$. Then S satisfies (K_1) and its right-left symmetry (K_r) by ([1]; Theorem 3.3). Thus by ([1], Theorem 2.2) every closed left (right resp.) ideal is an annihilator left (right resp.) ideal. Therefore $L(S)$ is dually isomorphic to $R(S)$, and so $\dim L(S) = \dim R(S)$. By Theorem 3 and its right-left symmetry it follows that S fulfills (A_1) and (A_r).

Conversely, suppose that S satisfies (A_1) and (A_r). Since $L(S)$ is finite dimensional, $S = X_1 \cup \cdots \cup X_n$ where each X_i is an atom of $L(S)$ and $n = \dim L(S)$. Then $0 = r(S) = r(X_1) \cap \cdots \cap r(X_n)$, where each $r(X_i)$ is a dual atom of $R(S)$. Hence $\dim R(S) \leq n$. Similarly $\dim R(S) \geq n$, and therefore $\dim R(S) = n = \dim L(S)$. By Theorem 3 and ([1]; Theorem 3.3) it follows that $\bar{S}_1 = \bar{S}_r$, completing the proof.

Reference

[1] Yuzo Utumi: On rings of which any one-sided quotient rings are two-sided, Proc. Amer. Math. Soc., **14**, 141–147 (1963).