60. A Remark on Gentzen's Paper "Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie". I

By Gaisi TAKEUTI

Department of Mathematics, Tokyo University of Education, Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., May 11, 1963)

In his paper [2], G. Gentzen proved the following theorem:

The transfinite induction up to the first ε-number ε_0 is not provable in the theory of natural numbers (formalized in the first order predicate calculus), while the transfinite induction up to an arbitrary ordinal number less than ε_0 is provable in the theory of natural numbers.

For the proof he introduced ordinal numbers up to ε_0 as individual constants of the system. But, following his method, the theorem can be stated in a more general form. That is, the linear ordering which is to be proved as a well-ordering in the system may be any general recursive linear ordering. The purpose of this paper is to remark this fact and state the following theorem in several cases. We shall begin with defining some notions and notations.

Let S be a (constructively defined) set which is well-ordered by a relation $<^*$. For any element s of S, $|s|$ stands for the order-type represented by s in the sense of $<^*$. We shall call $|s|$ the *value* of s in the sense of $<^*$. By $|S|$ we shall denote the least ordinal number α such that $|s|<\alpha$ for every $s\in S$. $|S|$ is called the *value* of S.

Let $\mathfrak{S}$ be a theory of natural numbers (formalized in the first or second order predicate calculus of Gentzen's style (cf. [1])). Throughout this paper we allow every sequence $\Gamma\rightarrow\Delta$ with the following properties as a mathematische Grundsequenz; every formula consisting of Γ or Δ is general recursive and containing no logical symbol and every sequence obtained from $\Gamma\rightarrow\Delta$ by replacing all free variables in Γ, Δ by arbitrary numerals (terms of the form $0'^{\cdots\prime}$) is true. The formal consistency of every consistent axiomatizable system can be proved in $\mathfrak{S}$. For this reason our result of this paper does not follow from Gödel's incompleteness theorem.

Let $P(a)$ be a general recursive predicate (containing no logical symbol) and $a\prec b$ be a linear ordering of the set $\{a\mid P(a)\}$. If it is provable in $\mathfrak{S}$ that $a\prec b$ is a well-ordering of the set, we call $a\prec b$ a *provable well-ordering* in $\mathfrak{S}$.

Theorem. *Let $\mathfrak{S}$ be a theory of natural numbers and S a con-*

structively defined set which is well-ordered by $<^*$. Moreover, let the consistency of $\mathfrak{S}$ be proved by using the transfinite induction on S in the same way as in [1] or [7]. The order-type of any provable well-ordering in $\mathfrak{S}$ is less than the value of S.

For many of such pairs $\mathfrak{S}$ and S, the following result (†) is also true:

(†) *Any well-ordering $\prec$ whose order-type is less than $|S|$ is a provable well-ordering in $\mathfrak{S}$.*

By the theorem and the result of [4], the least ordinal number not representable in Ackermann's system of ordinal numbers is less than the least ordinal number not representable by ordinal diagrams of order 2. We have not yet known the direct proof for this. Moreover, since the consistency of a formal theory of Schütte's ordinal numbers [5] can be proved by using the transfinite induction on ordinal diagrams of order 2 in the same way as in [4], the least ordinal number not representable in Schütte's system of ordinal numbers is less than the least ordinal number not representable by ordinal diagrams of order 2.

In the following we shall consider systems $\mathfrak{S}_0$-$\mathfrak{S}_3$ obtained from LK ([1]) or subsystems of G^1LC ([6]). For $i (i \leq 3)$, $P_i(a)$ and $a \prec_i b$ will denote general recursive predicates, the latter of which is a linear ordering of the set $\{a | P_i(a)\}$. We shall denote by $|a|_i$ the order-type represented by a natural number a such that $P_i(a)$, i.e. if $\{b | b \prec_i a\}$ is well-ordered, $|a|_i$ is an informal ordinal number.

$\mathcal{E}(*)$ is used as a free variable for predicate.

TJ-proof-figure with respect to $\mathfrak{S}_i$ (abbr. *TJ_i-proof-figure*) is defined to be a figure which is obtained from a proof-figure of $\mathfrak{S}_i$ modifying it as follows:

1. The following sequence called *TJ-upper sequence* with respect to $\mathfrak{S}_i$ (abbr. TJ_i-upper sequence) is allowed as a beginning sequence as well as beginning sequences of $\mathfrak{S}_i$:

$$e_i \prec_i r,\ \forall x((\neg x \prec_i r) \vee \mathcal{E}(x)) \to \mathcal{E}(r) \qquad \text{for } i=0$$
$$P_i(r),\ \forall x(x \prec_i r \vdash \mathcal{E}(x)) \to \mathcal{E}(r) \qquad \text{for } 0<i\leq 3$$

where e_i stands for the first element with respect to $\prec_i$ and r is an arbitrary term (though Gentzen restricted it as a 'Zahlterm').

2. The following inference 'replacement of a term' is added:

$$\frac{\Gamma_1, A(s), \Gamma_2 \to \Delta}{\Gamma_1, A(t), \Gamma_2 \to \Delta} \quad \text{or} \quad \frac{\Gamma \to \Delta_1, A(s), \Delta_2}{\Gamma \to \Delta_1, A(t), \Delta_2}$$

where s and t are terms containing no variable and stand for the same numeral.

3. The end sequence is of the form

$$\mathcal{E}(e_i) \to \mathcal{E}(s_1), \cdots, \mathcal{E}(s_n) \qquad \text{for } i=0$$
$$\to \mathcal{E}(s_1), \cdots, \mathcal{E}(s_n) \qquad \text{for } 0<i\leq 3$$

where $s_1, \cdots, s_n$ are numerals.

Let $\mathfrak{P}$ be a TJ-proof-figure the right side of whose end sequence is $\mathcal{E}(s_1), \cdots, \mathcal{E}(s_n)$. The *end number* of $\mathfrak{P}$ is the minimum of $|s_1|_i, \cdots, |s_n|_i$. In what follows we assign TJ_i-proof-figure an element of a well-ordered set. If such an element σ is assigned to $\mathfrak{P}$, the value $|\sigma|$ is called a *value* of $\mathfrak{P}$.

The system $\mathfrak{S}_0$. $\mathfrak{S}_0$ is obtained from LK modifying it as follows:

1. Every beginning sequence of $\mathfrak{S}_0$ is of the form $D \rightarrow D$ or of the form $a=b$, $A(a) \rightarrow A(b)$ or the 'mathematische Grundsequenz' in the above sense.

2. The following inference-schema called 'induction' (VJ-Schlußfigur) is added:

$$\frac{A(a),\ \Gamma \rightarrow \Delta,\ A(a')}{A(0),\ \Gamma \rightarrow \Delta,\ A(t)}$$

where a is contained in none of $A(0)$, Γ, Δ and t is an arbitrary term. $A(a)$ and $A(a')$ are called the *chief-formulas* (Hauptformeln) and a is called an *eigenvariable* of this induction.

The consistency of $\mathfrak{S}_0$ is proved by using the transfinite induction up to ε_0 (cf. [**1**]).

Theorem 0. *Let $\mathcal{E}(a)$ be an arbitrary formula in $\mathfrak{S}_0$. If the sequence*

$$e_0 \prec_0 a,\ Vx(P_0(x) \wedge Vy(y \prec_0 x \vdash \mathcal{E}(y)) \vdash \mathcal{E}(x)) \rightarrow \mathcal{E}(a)$$

is provable in $\mathfrak{S}_0$, the order-type of $\prec_0$ is less than ε_0.

$(\dagger)_0$ *Any well-ordering whose order-type is less than ε_0 is a provable well-ordering in $\mathfrak{S}_0$.*

Let m be a numeral and $\rightarrow P_0(m)$ a mathematische Grundsequenz. Under the assumption of the theorem, we have a TJ_0-proof-figure with the end number $|m|_0$. For the proof of the theorem we have only to prove that, in the same method as in [**2**], we can assign every TJ_0-proof-figure an ordinal number less than ε_0 as a 'value' and define the reduction for every proof-figure whose end number is not 0. This can be done by the same assignment of ordinal numbers and the same reduction in [**2**]. $(\dagger)_0$ can be proved in the same way as in §2 of [**2**].

In the following we shall consider some modified systems of G^1LC.

Let $1 \leq i \leq 3$. The precise definition of $\mathfrak{S}_i$ will be given below. Let S_i be a constructively defined and well-ordered set such that the consistency of $\mathfrak{S}_i$ is proved by using the transfinite induction on S_i. Let $J_i(a)$ be the abbreviation of

$$V\varphi(P_i(a) \wedge Vx(P_i(x) \wedge Vy(y \prec_i x \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[a]).$$

Our theorem states:

If $P_i(a) \rightarrow J_i(a)$ is provable in $\mathfrak{S}_i$, then the order-type of $\prec_i$ is *less than the value of* $\mathfrak{S}_i$.

LEMMA 1. *Let* $P_i(a)\to J_i(a)$ *be provable in* $\mathfrak{S}_i$. *For every* n *such that the sequence* $\to P_i(n)$ *is a mathematische Grundsequenz, we have a* TJ_i*-proof-figure with the end number* $|n|_i$.

In the following we shall define the system $\mathfrak{S}_i$ exactly and give the outline of the proof.

Definition of the system $\mathfrak{S}_1$. $\mathfrak{S}_1$ is a system obtained from G^1LC modifying it as follows:

1. Every beginning sequence of $\mathfrak{S}_1$ is of the form $D\to D$ or the form $a=b,\ A(a)\to A(b)$ or a 'mathematische Grundsequenz' in the sense of this paper.
2. The inference-schema 'induction' is added.
3. The inference V left on an f-variable of the form

$$\frac{F(V),\ \Gamma\to\Delta}{V\varphi F(\varphi),\ \Gamma\to\Delta}$$

is restricted by the condition that $V\varphi F(\varphi)$ is regular. The definition of a regular formula is seen in §2 of [7]. Roughly speaking, a formula A is regular if the following condition is satisfied: Let $\#$, $\natural$ be any pair of proper V's on f-variables in A with the forms $\#\varphi B(\psi)$ and $\natural\varphi C(\varphi)$, respectively. If $\natural\varphi C(\varphi)$ appears in $B(\psi)$ and $\natural$ is negative to $\#$, then $\natural$ is isolated, where '$\natural$ is isolated' means: (i) $C(\varphi)$ contains no free f-variable; (ii) φ is not contained in the scope of any V on an f-variable contained in $C(\varphi)$; (iii) if $\natural\varphi C(\varphi)$ appears in $V\eta D(\eta)$ in A, $C(\varphi)$ contains no η.

The consistency of $\mathfrak{S}_1$ is proved by using the transfinite induction of ordinal diagrams of order n (cf. [7]). We shall define the reduction of TJ_1-proof-figures whose end numbers are not 0.

We define 'isolated degree' as in [7] and introduce the inference 'substitution' satisfying the conditions 2.5, Chapter 1 of [7] in $\mathfrak{S}_1$. Moreover, we assign the same ordinal diagram of order n to every sequence of a proof-figure of $\mathfrak{S}_1$ as in the case of RNN and assign

$$(1\,;1,\ (1\,;1,\ (1\,;1,\ (1\,;1,\ 1\#\ (1\,;1,\ 1)))))$$

to a TJ_1-upper sequence. (We regard $A\vdash B$ as $\neg(A\wedge\neg B)$.) We define 'end-place' as the 'Endstück' in [2].

We are to prove that the end number of any TJ_1-proof-figure is not greater than the value of the TJ_1-proof-figure. We can follow the reduction in [7] till we face a TJ_1-proof-figure $\mathfrak{P}$ with the following properties:

p1. The end-place of $\mathfrak{P}$ contains no free variable.

p2. $\mathfrak{P}$ contains no 'induction' as an inference of the boundary.

p3. The end-place of $\mathfrak{P}$ contains no beginning sequence for equality.

p 4. The end-place of $\mathfrak{P}$ contains no beginning sequence of the form $D \rightarrow D$.

p 5. If the end-place of $\mathfrak{P}$ contains a weakening $\mathfrak{J}$, it contains no other inference than weakenings under $\mathfrak{J}$.

LEMMA 2. *If a TJ_1-proof-figure has the above properties* p 1–5, *it must contain at least one logical inference or TJ_1-upper sequence.*

LEMMA 3. *Let $\mathfrak{P}$ be a TJ_1-proof-figure having the properties* p 1–5. *If the 'Verknüpfungsreduktion' cannot be defined* (*i.e.* $\mathfrak{P}$ *contains no suitable cut*), $\mathfrak{P}$ *contains a TJ_1-upper sequence in its end-place.*

A TJ_1-proof-figure is called 'critical', if it has properties p1–5 and

p 6. $\mathfrak{P}$ contains no suitable cut.

LEMMA 4. *Let $\mathfrak{P}$ be a critical TJ_1-proof-figure. Then there is an end-formula such that it belongs to the same string* (*Bund*) *with a formula in the right side of a TJ_1-upper sequence in the end-place.*

By means of Lemmas 3 and 4 we can define the 'kritische Reduktion' in the same way as in [**2**], so that a critical TJ_1-proof-figure can be reduced to a TJ_1-proof-figure with any end number less than the end number of the critical TJ_1-proof-figure.

LEMMA 5. *For every TJ_1-proof-figure, the end number is not greater than the value.*

From Lemma 5 we can complete the proof of the theorem. (†) for this case can be seen from [**10**].

The definition of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and the proof of the theorem for these systems can be seen in the second paper of this title.

(*To be continued*)