75. A Generalization of König's Lemma

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König proved in [1] the following lemma:

Let $E_1, E_2, \cdots, E_n, \cdots$ be an enumerable sequence of finite and non empty sets and R a relation of two arguments satisfying the following condition: for every element x_{n+1} of E_{n+1} $(n \geq 0)$ there is an element x_n of E_n corresponding to x_{n+1} by the relation R i.e. $x_n R x_{n+1}$. *Then we can obtain an infinite sequence $a_1, a_2, \cdots, a_n, \cdots$ such that $a_n \in E_n$ and $a_n R a_{n+1}$ $(n=1, 2, \cdots)$.*

Sometimes, it is also called Brouwer's fan theorem.

In this paper we shall prove a generalization of this lemma. Let R be a set. p stands for the element of R. A finite set W_p is assigned for every $p \in R$. If R_1 is a subset of R and f is an element of $\prod_{p \in R_1} W_p$, then f is called a *partial function* (over R) and $D(f)$ (the domain of f) is defined to be R_1. If $D(f)=R$, then f is called a *total function*. If f and g are partial functions and $D(f)=D_0 \subseteq D(g)$ and $f(x)=g(x)$ for every $x \in D_0$, then we call g an *extension* of f and write $f \prec g$ and $f=g \upharpoonright D_0$. If $f \prec g$ and $D(g)=D(f) \cup N$, then we say 'g is an extension of f over N'.

THEOREM. *Let P be a property about partial function satisfying the following conditions:*

1. *$P(f)$ holds if and only if there exists a finite subset N of R satisfying $P(f \upharpoonright N)$.*
2. *$P(f)$ holds for every total function f.*

Then there exists a finite subset N_0 of R such that $P(f)$ holds if $D(f) \supseteq N_0$.

It is noted that $\overline{\overline{R}}$ be arbitrary large. The case that R is the set of natural numbers is the original König's lemma.

To prove this theorem we shall first define several concepts. We say $\widetilde{P}(f)$ if there exists a finite subset N of R such that every extension of f over N satisfies P. Clearly $\widetilde{P}(f)$ holds for every total function.

We define $f*g$ to be the function uniquely defined by the following conditions:

1) $D(f*g)=D(f) \cup D(g)$
2) $f \prec f*g$
3) If $p \in D(g)-D(f)$, then $(f*g)(p)=g(p)$.

If g is the function whose domain is $\{p\}$ and $g(p)=w_p$, then we use sometimes the abbreviated notation $f*w_p$ for $f*g$.

LEMMA 1. *If* $\widetilde{P}(f*w_p)$ *for every* $w_p \in W_p$ *then* $\widetilde{P}(f)$, *and vice versa.*

PROOF. Since the sufficiency is clear, we shall assume $\widetilde{P}(f*w_p)$ for every $w_p \in W_p$. Let $w_p^1, \cdots, w_p^n$ be the series of all the elements of W_p. For every $i\,(1 \leq i \leq n)$ there exists a finite subset N^i of R such that every extension of $f*w_p^i$ over N^i satisfies P. N is defined to be $N^1 \cup \cdots \cup N^n \cup \{p\}$. We see clearly that every extension of f over N satisfies P.

PROOF of Theorem. We have only to prove that the empty function ϕ satisfies $\widetilde{P}$. We shall assume $\neg\widetilde{P}(\phi)$ ($\neg\widetilde{P}(\phi)$ is read '$\widetilde{P}(\phi)$ does not hold'). A family $\mathfrak{F}$ of partial functions is called *regular* if the following conditions are fulfilled.

1) If $f \in \mathfrak{F}$ and $g \in \mathfrak{F}$, then either $f \prec g$ or $g \prec f$ holds.

2) If $f \in \mathfrak{F}$, then $\neg\widetilde{P}(f)$.

In virtue of Zorn's lemma, there exists a maximal regular family $\mathfrak{F}_0$.

If $\mathfrak{F}_0$ contains a maximal element f_0, then $\widetilde{P}(f_0)$ does not hold whence follows $D(f_0) \subsetneqq R$. In this case, we can extend f_0 to a function g satisfying $\neg\widetilde{P}(g)$ by using Lemma 1. This is a contradiction. Thus $\mathfrak{F}_0$ does not contain a maximal element.

There exists uniquely a function f_0 satisfying the following conditions.

1) $D(f_0) = \bigcup_{f \in \mathfrak{F}_0} D(f)$.

2) $f \prec f_0$ holds for every $f \in \mathfrak{F}_0$.

If $\neg\widetilde{P}(f_0)$, then $f_0 \in \mathfrak{F}_0$ and f_0 is the maximal element of $\mathfrak{F}_0$, whence follows a contradiction. Therefore $\widetilde{P}(f_0)$ holds. Hence there exists a finite subset N of R such that every extension of f_0 over N satisfies P. Let $g_1, \cdots, g_n$ be the series of all the functions whose domain is N. Since $P(f_0*g_i)$ for every $i\,(1 \leq i \leq n)$, there exists $h_i \prec f_0$ such that $P(h_i*g_i)$ and $D(h_i)$ is a finite subset. Therefore $D(h_1*\cdots*h_n)$ is finite. Let $p_1, \cdots, p_m$ be the series of all the points of $D(h_1*\cdots*h_n)$. There exists a function $f_i \in \mathfrak{F}_0$ whose domain contains $p_i (1 \leq i \leq m)$. Let f_k be the maximum among $f_1, \cdots, f_m$. Then $h_1*\cdots*h_n \prec f_k$. Hence follows $\widetilde{P}(f_k)$ which contradicts $f_k \in \mathfrak{F}_0$.

Reference

[1] D. König: Sur les correspondances multivoque des ensembles, Fund. Math., **8**, 114–134 (1926).