104. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. VIII

By Sakuji INOUE
Faculty of Education, Kumamoto University

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On the assumption that $S(\lambda)$ and $R(\lambda)$ are the functions defined in the statement of Theorem 1 [cf. Proc. Japan Acad., Vol. 38, 263–268 (1962)], in the preceding papers we have discussed, under some conditions, the distribution of ζ-points of those functions in the exterior of a suitably large circle with center at the origin and the relation between the two finite exceptional values of those functions for the exterior of that circle, by using the extended Fourier series expansion of the function $\chi(\lambda)$ defined as the sum of the two principal parts of $S(\lambda)$. In the present paper, however, we shall treat, under some conditions, those problems with respect to the derivatives of $R(\lambda)$ and $S(\lambda)$ from a different point of view, by applying the integral expressions of the derivatives of $\chi(\lambda)$.

Theorem 21. Let $S(\lambda)$ and $\{\lambda_\nu\}$ be the same notations as those defined in the statement of Theorem 1; let $R(\lambda)$ be the ordinary part of $S(\lambda)$; let σ be a given positive constant satisfying the inequality $\sup_\nu |\lambda_\nu| < \sigma < \infty$; let $\{z_n\}$ be a set of mutually distinct ζ-points of $R(\lambda)$ such that

$$\left.\begin{array}{l} R(z_n)=\zeta \\ \sigma<|z_n|\leqq|z_{n+1}| \end{array}\right\} \ (n=1,2,3,\cdots),\quad |z_n|\to\infty \ (n\to\infty);$$

let

$$\text{(A)}\qquad \frac{R^{(m)}(0)}{R^{(m-1)}(0)}\neq c,\quad \frac{R^{(m+\mu)}(0)}{R^{(m+\mu-1)}(0)}=c \ (\mu=1,2,3,\cdots),$$

where c is a non-zero complex constant and m is a positive integer; let r_n, $n=1,2,3,\cdots$, be positive numbers satisfying the conditions

$$\text{(B)}\qquad \left|\frac{R(z_n+r_ne^{i\theta})-R(z_n)}{r_ne^{i\theta}}-R'(z_n)\right|<\varepsilon,\quad \inf_n\{r_n|z_n|^{m+1-\varepsilon}\}\neq 0$$

for a given positive number ε less than 1; let $\widetilde{R}_\alpha(\lambda)=R(\lambda)-P_\alpha(\lambda)$ and $\widetilde{S}_\alpha(\lambda)=S(\lambda)-P_\alpha(\lambda)$, where

$$P_\alpha(\lambda)=\sum_{j=\alpha}^{m}\frac{1}{j!}\{R^{(j)}(0)-cR^{(j-1)}(0)\}\lambda^j \ (\alpha=1,2);$$

and let Γ_n denote the circle $|\lambda-z_n|=r_n$. Then any z_n in the set $\{z_n\}$ is a $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-point of $\widetilde{R}'_\alpha(\lambda)$ and there exists a suitably large positive integer L such that in the interior of each of the circles Γ_{L+p}, $p=0,1,2,\cdots$, $\widetilde{S}'_\alpha(\lambda)$ has $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-points the number of which equals that of $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-

points of $\widetilde{R}'_\alpha(\lambda)$ in the interior of the same circle as it; and moreover the multiplicity of any $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-point of each of the functions $\widetilde{R}'_\alpha(\lambda)$ and $\widetilde{S}'_\alpha(\lambda)$ is equal to 1, as far as the $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-point lies in the domain $\{\lambda: |\lambda|\geqq|z_L|\}$.

Proof. By applying the hypothesis (A) to Taylor's power series expansion of $R(\lambda)$ or to the extended Fourier series expansion of $R(\lambda)$ which was shown in Part III [cf. Proc. Japan Acad., Vol. 38, 641–645 (1962)], we can first show without difficulty that

$$R'(\lambda)=\sum_{j=1}^{m}\frac{1}{(j-1)!}\{R^{(j)}(0)-cR^{(j-1)}(0)\}\lambda^{j-1}+cR(\lambda),$$

where $0!$ and $R^{(0)}(0)$ denote 1 and $R(0)$ respectively, so that

$$\begin{aligned}\widetilde{R}'_\alpha(\lambda)&=R'(\lambda)-P'_\alpha(\lambda)\\ &=(\alpha-1)[R'(0)-cR(0)]+cR(\lambda).\end{aligned}\tag{17}$$

Since, by hypotheses, z_n, $n=1,2,3,\cdots$, are ζ-points of $R(\lambda)$, it turns out at once from (17) that $\widetilde{R}'_\alpha(\lambda)$ has z_n, $n=1,2,3,\cdots$, as $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-points of itself. On the other hand, by virtue of the application of (17) and the first condition in (B) we have

$$\begin{aligned}&|\widetilde{R}'_\alpha(z_n+r_ne^{i\theta})-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}|\\ &\quad=|c|\,|R(z_n+r_ne^{i\theta})-R(z_n)|\\ &\quad>|c|r_n(|R'(z_n)|-\varepsilon)\\ &\quad\geqq|c|r_n\{|P'_\alpha(z_n)|-(\alpha-1)|R'(0)-cR(0)|-|c\zeta|-\varepsilon\},\end{aligned}\tag{18}$$

where $P'_\alpha(z_n)$ is a polynomial in z_n of the degree $m-1$ by virtue of the hypothesis $R^{(m)}(0)-cR^{(m-1)}(0)\neq 0$ in (A) and hence the right-hand side of (18) is positive for all suitably large values of n. Furthermore, if we denote by $\chi(\lambda)$ the sum of the first and the second principal parts of $S(\lambda)$ and put

$$c_\mu=\frac{1}{\pi}\int_0^{2\pi}S(\rho e^{it})e^{i\mu t}dt\ \ (\sup_\nu|\lambda_\nu|<\rho<\sigma,\ \mu=0,\pm1,\pm2,\cdots),$$

then, as we have already shown in Part III,

$$\chi\left(\frac{\rho}{\kappa}e^{i\theta}\right)=\frac{1}{2}\sum_{\mu=1}^{\infty}c_\mu\left(\frac{\kappa}{e^{i\theta}}\right)^\mu\ \ (0<\kappa<1),$$

where the right-hand member is an absolutely and uniformly convergent series of $e^{-i\theta}$, so that

$$\chi'\left(\frac{\rho}{\kappa}e^{i\theta}\right)=-\frac{\kappa}{2\rho e^{i\theta}}\sum_{\mu=1}^{\infty}\mu c_\mu\left(\frac{\kappa}{e^{i\theta}}\right)^\mu.$$

It follows immediately from this result that

$$\begin{aligned}\chi'(z_n+r_ne^{i\theta})&=-\frac{1}{2(z_n+r_ne^{i\theta})}\sum_{\mu=1}^{\infty}\mu c_\mu\left(\frac{\rho}{z_n+r_ne^{i\theta}}\right)^\mu\\ &=-\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{\rho e^{it}}{\left(1-\dfrac{\rho e^{it}}{z_n+r_ne^{i\theta}}\right)^2}dt\cdot\frac{1}{(z_n+r_ne^{i\theta})^2}\end{aligned}$$

$$=-\frac{1}{2\pi}\int_0^{2\pi} S(\rho e^{it})\frac{\rho e^{it}}{(z_n+r_ne^{i\theta}-\rho e^{it})^2}dt$$

for every n with $|z_n+r_ne^{i\theta}|>\rho$. In addition, since the extended Fourier series expansion of $S\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ is rewritten in the form

$$S\left(\frac{\rho}{\kappa}e^{i\theta}\right)=\frac{1}{2}\sum_{\mu=-\infty}^{\infty}c_\mu\left(\frac{\kappa}{e^{i\theta}}\right)^\mu \quad (0<\kappa<1)$$

and since it can be verified by the familiar Parseval identity that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|S(\rho e^{it})|^2dt\right\}^{\frac{1}{2}}=\frac{1}{2}\left\{\sum_{\mu=-\infty}^{\infty}|c_\mu|^2\right\}^{\frac{1}{2}}\equiv K(\rho),$$

an application of Schwarz's inequality to the just established relation yields the inequality

$$(19)\qquad |\chi'(z_n+r_ne^{i\theta})|\leqq\frac{\rho K(\rho)}{(|z_n|-r_n-\rho)^2}$$

holding for every n with $|z_n|>r_n+\rho$.

On combining (18) and (19), we have

$$\frac{|\widetilde{R}'_\alpha(z_n+r_ne^{i\theta})-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}|}{|\chi'(z_n+r_ne^{i\theta})|}$$
$$>\frac{(|z_n|-r_n-\rho)^2|c|r_n\{|P'_\alpha(z_n)|-(\alpha-1)|R'(0)-cR(0)|-|c\zeta|-\varepsilon\}}{\rho K(\rho)}$$

for every z_n with modulus greater than $r_n+\rho$. Since, by the hypothesis $\inf_n\{r_n|z_n|^{m+1-\varepsilon}\}\neq 0$ in (B), the numerator of the fraction on the right-hand side of the last inequality becomes infinite with n, there exists a suitably large positive integer L such that

$$(20)\qquad |\widetilde{R}'_\alpha(z_{L+p}+r_{L+}e_p{}^{i\theta})-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}|$$
$$>|\chi'(z_{L+p}+r_{L+p}e^{i\theta})|\quad (p=0,1,2,\cdots).$$

Moreover it can be found from the equality $S(\lambda)=R(\lambda)+\chi(\lambda)$ that

$\widetilde{S}'_\alpha(\lambda)-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}=\{\widetilde{R}'_\alpha(\lambda)-[(\alpha-1)(R'(0)-cR(0))+c\zeta]\}+\chi'(\lambda)$, where $\chi'(\lambda)$ is regular inside and on each of the circles Γ_{L+p}, $p=0, 1, 2,\cdots$, because of the fact that $|z_{L+p}|>r_{L+p}+\rho$ and so also is $\widetilde{R}'_\alpha(\lambda)$ because of the fact that clearly $\widetilde{R}'_\alpha(\lambda)$ is a transcendental integral function. On the other hand, z_n, $n=1,2,3,\cdots$, are $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-points of $\widetilde{R}'_\alpha(\lambda)$ as we have already shown at the beginning of the proof of the present theorem.

An application of the Rouché theorem to (20) and the results just pointed out leads us to the conclusion that in the interior of each of the circles Γ_{L+p}, $p=0,1,2\cdots$, $\widetilde{S}'_\alpha(\lambda)$ has $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-points the number of which equals that of $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-points of $\widetilde{R}'_\alpha(\lambda)$ in the interior of the same circle as it. Moreover, since $\widetilde{R}'_\alpha(\lambda)-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}=c\{R(\lambda)-\zeta\}$ and since, as can be seen by means of the relation $R'(z_{L+p})=P'_\alpha(z_{L+p})+(\alpha-1)[R'(0)-cR(0)]+c\zeta$ deduced from (17) and of the inequality $|P'_\alpha(z_{L+p})|-(\alpha-1)|R'(0)-cR(0)|-|c\zeta|>\varepsilon$ shown before, $R'(z_{L+p})$ never vanishes for any value of $p=0,1,2,\cdots$, $\widetilde{R}'_\alpha(\lambda)-\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$ has z_{L+p} as its zero-point with multiplicity 1. Consequently the multiplicity of any $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-point of each of the functions $\widetilde{R}'_\alpha(\lambda)$ and $\widetilde{S}'_\alpha(\lambda)$ is equal to 1, as far as the $\{(\alpha-1)[R'(0)-cR(0)]+c\zeta\}$-point lies in the domain $\{\lambda: |\lambda|\geqq|z_L|\}$.

With these results the proof of the present theorem is complete.

Corollary 3. Let $\zeta\neq R(0)-\dfrac{R'(0)}{c}$ and $m=1$ in Theorem 21; let ε be an arbitrarily given positive number less than $|R'(0)-cR(0)+c\zeta|$, not zero; let r be a positive number such that

$$\text{(C)}\qquad \left|\frac{R(z_n+re^{i\theta})-R(z_n)}{re^{i\theta}}-R'(z_n)\right|<\varepsilon\ (n=1,2,3,\cdots)$$

(it will be shown later on that in this case there exist in fact such many r irrespective of n); let L be the least value of n satisfying the condition

$$\text{(D)}\qquad |z_n|>\sqrt{\frac{\rho K(\rho)}{|c|r[|R'(0)-cR(0)+c\zeta|-\varepsilon]}}+r+\rho,\ (\sup_\nu|\lambda_\nu|<\rho),$$

where $K(\rho)=\left\{\sum_{\mu=-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{it})e^{i\mu t}dt\right|^2\right\}^{\frac{1}{2}}(<\infty)$; and let Γ_p denote the circle $|\lambda-z_{L+p}|=r$ for each value of $p=0,1,2,\cdots$. Then z_n, $n=1,2,3,\cdots$, are $[R'(0)-cR(0)+c\zeta]$-points of $R'(\lambda)$, and in the interior of any circle Γ_p $S'(\lambda)$ has $[R'(0)-cR(0)+c\zeta]$-points the number of which equals that of $[R'(0)-cR(0)+c\zeta]$-points of $R'(\lambda)$ in the interior of the same circle Γ_p; and in addition, the multiplicity of any ζ-point z_n of $R(\lambda)$ is equal to 1, and $R(0)-\dfrac{R'(0)}{c}$ is the exceptional value of $R(\lambda)$.

Proof. Since, by hypotheses, $m=1$, we have

$$\text{(21)}\qquad R'(\lambda)=R'(0)-cR(0)+cR(\lambda),$$

as can be found from the first relation shown at the beginning of the proof of Theorem 21, and (21) implies that every z_n is a $[R'(0)-cR(0)+c\zeta]$-point of $R'(\lambda)$ and that the multiplicity of z_n as a ζ-point of $R(\lambda)$ equals 1 because of the fact that $R'(z_n)$ is never zero by the hypothesis $\zeta\neq R(0)-\dfrac{R'(0)}{c}$. Moreover it is a matter of simple manipulations to show by (21) and the hypothesis $\dfrac{R^{(\mu+1)}(0)}{R^{(\mu)}(0)}=c$, $\mu=1,2,3,\cdots$, that

$$\text{(22)}\qquad R(\lambda)=R(0)-\frac{R'(0)}{c}+\frac{R'(0)}{c}e^{c\lambda}.$$

Accordingly $R(0)-\dfrac{R'(0)}{c}$ is the exceptional value of $R(\lambda)$, as we were to prove.

Since it is easily verified by direct computation from (21) and (22) that

$$\frac{R(z_n+re^{i\theta})-R(z_n)}{re^{i\theta}}-R'(z_n)=[R'(0)-cR(0)+c\zeta]cre^{i\theta}\sum_{\mu=2}^{\infty}\frac{(cre^{i\theta})^{\mu-2}}{\mu!},$$

there exist in fact positive numbers r satisfying (C) for all values of n; and hence, by means of (21), (C), and the hypotheses concerning ζ and ε, we obtain

$$\begin{aligned}|R'(z_n+re^{i\theta})-[R'(0)-cR(0)+c\zeta]| &= |c|\,|R(z_n+re^{i\theta})-R(z_n)| \\ &> |c|r[\,|R'(0)-cR(0)+c\zeta|-\varepsilon]>0\end{aligned}$$

for such an r. Furthermore, by reference to (19) and this inequality, we have

$$\begin{aligned}&\frac{|R'(z_n+re^{i\theta})-[R'(0)-cR(0)+c\zeta]|}{|\chi'(z_n+re^{i\theta})|} \\ &\qquad > \frac{(|z_n|-r-\rho)^2|c|r[\,|R'(0)-cR(0)+c\zeta|-\varepsilon]}{\rho K(\rho)}\end{aligned}$$

for every z_n with modulus greater than $r+\rho$, so that, by the hypothesis concerning L,

$$|R'(z_{L+p}+re^{i\theta})-[R'(0)-cR(0)+c\zeta]|>|\chi'(z_{L+p}+re^{i\theta})| \quad (p=0,1,2,\cdots).$$

In consequence, the Rouché theorem and the same reasoning as that used in the course of the proof of Theorem 21 permit us to assert that in the interior of any circle Γ_p $S'(\lambda)$ has $[R'(0)-cR(0)+c\zeta]$-points the number of which equals that of $[R'(0)-cR(0)+c\zeta]$-points of $R'(\lambda)$ in the interior of the same circle Γ_p, as we wished to prove.

Theorem 22. Let $S(\lambda)$, $\{\lambda_\nu\}$, σ, and ρ be the same notations as before; let $R(\lambda)$ be the ordinary part of $S(\lambda)$; let m be a positive integer; let $\{z_n\}$ be a set of mutually distinct ζ-points of $R^{(m-1)}(\lambda)$ such that $\sigma<|z_n|\leqq|z_{n+1}|$ for every positive integer n and $|z_n|\to\infty$ $(n\to\infty)$; let $d=\inf_n|R^{(m)}(z_n)|>0$; let r be a positive number satisfying the condition

$$\left|\frac{R^{(m-1)}(z_n+re^{i\theta})-R^{(m-1)}(z_n)}{re^{i\theta}}-R^{(m)}(z_n)\right|<\varepsilon \quad (n=1,2,3,\cdots)$$

for a given positive number ε less than d; and let L be the least value of n such that

$$\text{(E)} \qquad |z_n|>\sqrt[m]{\frac{(m-1)!\,\rho K(\rho)}{r(d-\varepsilon)}}+r+\rho,$$

where $K(\rho)=\left\{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|S(\rho e^{it})|^2dt\right\}^{\frac{1}{2}}$. Then, in the interior of each circle $|\lambda-z_{L+p}|=r$, $p=0,1,2,\cdots$, $S^{(m-1)}(\lambda)$ has ζ-points the number of which equals that of ζ-points of $R^{(m-1)}(\lambda)$ in the interior of the same circle as it; and moreover, if the above hypotheses are satisfied for every $\zeta(\neq\infty)$ different from the finite exceptional value of $R^{(m-1)}(\lambda)$ and if ξ is the finite exceptional value of $S^{(m-1)}(\lambda)$ for the domain $\{\lambda: |\lambda|>\sigma\}$, then ξ is also the finite exceptional value of $R^{(m-1)}(\lambda)$.

Proof. Since, as can be seen from the expansion of $\chi\left(\frac{\rho e^{i\theta}}{\kappa}\right)$, $\chi(\lambda)=\frac{1}{2}\sum_{\mu=1}^{\infty}c_\mu\left(\frac{\rho}{\lambda}\right)^\mu$, $(|\lambda|>\rho)$, we can verify without difficulty from the termwise differentiability and integrability of a uniformly convergent series that

$$\chi^{(m-1)}(\lambda)=(-1)^{m-1}\frac{1}{2}\sum_{\mu=1}^{\infty}\mu(\mu+1)\cdots(\mu+m-2)c_\mu\left(\frac{\rho}{\lambda}\right)^\mu\frac{1}{\lambda^{m-1}}\quad(|\lambda|>\rho)$$

$$=(-1)^{m-1}\frac{(m-1)!}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{\frac{\rho e^{it}}{\lambda}}{\left(1-\frac{\rho e^{it}}{\lambda}\right)^m}dt\cdot\frac{1}{\lambda^{m-1}}$$

$$=(-1)^{m-1}\frac{(m-1)!}{2\pi}\int_0^{2\pi}S(\rho e^{it})\frac{\rho e^{it}}{(\lambda-\rho e^{it})^m}dt.$$

Hence, for every z_n with modulus greater than $r+\rho$,

$$|\chi^{(m-1)}(z_n+re^{i\theta})|\leqq\frac{(m-1)!\,\rho K(\rho)}{(|z_n|-r-\rho)^m}.$$

In consequence, we can establish without difficulty the former assertion of the present theorem by reasoning exactly like that used in the proof of Theorem 21. Moreover it is obvious that the latter assertion is a direct consequence of the former one.