131. *The Kernel Representation of the Fractional Power of the Elliptic Operator*

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1963)

§1. **Introduction.** Let A be a strongly elliptic operator defined in a domain D of R^n, and let us consider the Dirichlet problem for the operator $A+\lambda I$, λ be a complex number. Then we can define the fractional power $A^{-\alpha}$ under a suitable condition on the spectrum of A. In the case where A is formally self-adjoint, T. Kotake and M. S. Narasimhan [2] have recently proved that $A^{-\alpha}(\mathrm{Re}\,\alpha>0)$ has a kernel representation and moreover this kernel is very regular. In this article, we want to obtain the same result for not always self-adjoint operator. We consider the Dirichlet problem in the space $L^2(D)$. We express the weak solution $u\in L^2(D)$ of the equation $Au+\lambda u=f\in L^2(D)$ by means of parametrix according to H. G. Garnir [1], and we also express the Green kernel of $A+\lambda I$ using the Green operator G_λ. Finally, we show that the kernel $K^{(\alpha)}$ of $A^{-\alpha}$ is very regular. To show this, we used some properties of parametrix which are due to S. Mizohata [3]. The detailed proof will be given in a forthcoming paper.

I thank here Prof. Mizohata, who encouraged me in this subject.

§2. **Expression of solutions.** Let us consider the strongly elliptic partial differential operator of order $2m$ defined in a domain D (bounded or unbounded) of R^n

$$(2.1)\qquad A=A\left(x, \frac{\partial}{\partial x}\right)=\sum_{|\nu|\leqq 2m} a_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu, \text{ where}$$

$$\left(\frac{\partial}{\partial x}\right)^\nu=\left(\frac{\partial}{\partial x_1}\right)^{\nu_1}\left(\frac{\partial}{\partial x_2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\nu_n}.$$

The coefficients $a_\nu(x)$ belong to $\mathscr{B}(\widetilde{D})$, where $\widetilde{D}$ is an open set such that $\overline{D}\subseteq\widetilde{D}$. The condition of ellipticity

$$(2.2)\qquad \mathrm{Re}\sum_{|\nu|=2m} a_\nu(x)(iy)^\nu\geqq\gamma|y|^{2m}, \text{ for all } y\in R^n, \gamma\text{: const.}>0,$$

is to be fulfilled uniformly in D. We denote by $A'=A'\left(x, \frac{\partial}{\partial x}\right)$ the transposed operator of A. Because we only need the local expressions (expressions in a fixed compact set contained in D) of weak solutions and of Green kernels, without loss of generality we can suppose that the coefficients $a_\nu(x)$ are defined in R^n and the uniform ellipticity (2.2) holds in R^n as well.

At first, we assume the existence of the parametrix E of A (resp.

E' of A') having the following properties:
i) E (resp. E') satisfies the equation

$$A\left(x, \frac{\partial}{\partial x}\right)E(x, \xi)=\delta_{x-\xi}-L(x, \xi) \tag{2.3}$$

$$\left(\text{resp. } A'\left(x, \frac{\partial}{\partial x}\right)E'(x, \xi)=\delta_{x-\xi}-L'(x, \xi)\right)$$

where $L(x, \xi)$(resp. $L'(x, \xi)$) is a sufficiently smooth function of x in R^n, depending on the parameter ξ which runs through R^n.
ii) $E(x, \xi)$(resp. $E'(x, \xi)$) is semi-regular in x and ξ at the same time and is infinitely differentiable in (x, ξ) outside of the diagonal Δ, moreover, $E(x, \xi)\in\mathcal{B}_{x,\xi}(\omega)$ for any open set $\omega(\bar{\omega}\frown\Delta=\phi)$.

Let D_1 be any *bounded* open set such that $\bar{D}_1\subseteq D$, and

(2.4) $D_{1,\delta}=\{x\in D; \text{dis.}(x, D_1)<\delta\}$ where $0<\delta$ (fixed)$<\frac{1}{2}\text{dis.}(\partial D, D_1)$.

Define

(2.5) $\alpha_\delta(x)=\alpha_\delta(|x|)\in\mathcal{D}, \equiv 1$ for $|x|<\delta/2, \equiv 0$ for $|x|>\delta$.

(2.6) $\beta(x)\doteq\beta_{D_1}(x)\in\mathcal{D}(D), \equiv 1$ on a neighbourhood of $D_{1,\delta}$.

$$\Phi_\delta(x, \xi)=A\left(x, \frac{\partial}{\partial x}\right)[\{1-\alpha_\delta(x-\xi)\}E(x, \xi)]. \tag{2.7}$$

$$\Phi'_\delta(x, \xi)=A'\left(x, \frac{\partial}{\partial x}\right)[\{1-\alpha_\delta(x-\xi)\}E'(x, \xi)]. \tag{2.7$'$}$$

By virtue of the hypothesis ii), $\Phi_\delta(x, \xi)$(resp. $\Phi'_\delta(x, \xi))\in\mathcal{B}_{x,\xi}(R^n\times R^n)$.

From now on, we denote the integral over R^n or D by the dual form:

$$\langle f(x), g(x)\rangle_x=\int f(x)g(x)dx.$$

LEMMA. If $u(x)\in L^2(D)$ be a weak solution of the equation $Au(x)=f(x)\in L^2(D)$, the equality

(2.8) $u(\xi)=\langle\alpha_\delta(x-\xi)E'(x, \xi), f(x)\rangle_x+\langle\beta(x)\{\Phi'_\delta(x, \xi)+L'(x, \xi)\}, u(x)\rangle_x$

holds as a distribution in D_1. In the same way, the equality

(2.8)$'$ $v(\xi)=\langle\alpha_\delta(x-\xi)E(x, \xi), g(x)\rangle_x+\langle\beta(x)\{\Phi_\delta(x, \xi)+L(x, \xi)\}, v(x)\rangle_x$

holds in D_1 for a weak solution $v(x)\in L^2(D)$ of the transposed equation $A'v(x)=g(x)\in L^2(D)$ (c.f. [1], p. 66).

Next, the set of the complex numbers λ such that $A+\lambda I$ has its Green operator G_λ attached to the Dirichlet problem is an open set. If A satisfies the condition (2.2), there exists a real constant γ_1 such that, if $\text{Re}\,\lambda>\gamma_1$, G_λ exists and its restriction on $L^2(D)$ has the following estimate of the operator norm

$$\|G_\lambda\|_{\mathcal{L}(L^2(D), L^2(D))}\leqq\text{const.}/(\text{Re}\,\lambda-\gamma_1). \tag{2.9}$$

Moreover we impose the condition

(C) there exists no spectrum on the half-line $\lambda\geqq 0$.

In other words, there exists G_λ on the positive real axis. Such an assumption may be artificial, but it is essential for our definition of the fractional power in §3.

Using (2.8) and (2.8)′, we obtain the following

PROPOSITION 1. *If the operator A has the Green operator G, G has a kernel representation*

$$(Gf)(\xi)=\int_D K(\xi, x)f(x)dx$$

where

$$K(\xi, x)=\alpha_\delta(x-\xi)E'(x, \xi)+\langle\alpha_\delta(\eta-x)E(\eta, x),\beta(\eta)\{\Phi'_\delta(\eta, \xi)+L'(\eta, \xi)\}\rangle_\eta + \langle \underset{\zeta\to\eta}{G}[\beta(\zeta)\{\Phi_\delta(\zeta, x)+L(\zeta, x)\}], \beta(\eta)\{\Phi'_\delta(\eta, \xi)+L'(\eta, \xi)\}\rangle_\eta \tag{2.10}$$

in $(\xi, x)\in D_1\times D_1$ (this is an extension of the formula obtained in [1], p. 118).

Finally, let us apply the formula (2.10) to the operator $A+\lambda I$ (of course, we assume the properties of parametrix):

Let us denote $G_\lambda=\left\{A\left(x, \frac{\partial}{\partial x}\right)+\lambda I\right\}^{-1}$, $(G_\lambda f)(\xi)=\int_D K(\xi, x\,|\,\lambda)f(x)dx$, then we have*)

$$\begin{aligned}K(\xi, x\,|\,\lambda)&=\alpha_\delta(x-\xi)E'(x, \xi\,|\,\lambda)\\&+\langle\alpha_\delta(\eta-x)E(\eta, x\,|\,\lambda), \beta(\eta)\{\Phi'_\delta(\eta, \xi\,|\,\lambda)+L'(\eta, \xi\,|\,\lambda)\}\rangle_\eta\\&+\langle \underset{\zeta\to\eta}{G_\lambda}[\beta(\zeta)\{\Phi_\delta(\zeta, x\,|\,\lambda)+L(\zeta, x\,|\,\lambda)\}], \beta(\eta)\{\Phi'_\delta(\eta, \xi\,|\,\lambda)+L'(\eta, \xi\,|\,\lambda)\}\rangle_\eta.\end{aligned} \tag{2.11}$$

§3. **Fractional powers of A.** Under the condition (C) on the spectrum of A, we define the fractional power $A^{-\alpha}$ by the integral

$$A^{-\alpha}=\frac{1}{2\pi i}\int_\Gamma(-\lambda)^{-\alpha}(A+\lambda I)^{-1}d\lambda, \text{ for } \operatorname{Re}\alpha>0 \tag{3.1}$$

where the path Γ of integration consists of three parts: the positive real axis (from ∞ to $\rho(0<\rho\ll 1)$), the circle $|\lambda|=\rho$ (from $\lambda=\rho$ to $\lambda=\rho$ in the negative sense) and the positive real axis (from ρ to ∞). $(-\lambda)^{-\alpha}$ equals to its principal value on the negative real axis. By the estimate (2.9), the integral (3.1) converges and defines a continuous linear operator: $L^2(D)\to L^2(D)$.

PROPOSITION 2. *For any given non-negative integers p and q, we can construct a family of parametrix $E(x, \xi\,|\,\lambda)$ (continuously depending on $(\xi, \lambda)\in R^n\times\Gamma$) of the operator $A+\lambda I$ having the following properties:*

(1°) *$\lambda^q L(x, \xi\,|\,\lambda)$ remains bounded in $\mathscr{B}^p_{x,\xi}(R^n\times R^n)$ when λ tends to $+\infty$.*

(2°) *$\lambda^q E(x, \xi\,|\,\lambda)$ remains bounded (with respect to $\lambda\in\Gamma$) in $\mathscr{B}_{x,\xi}(\omega)$ where ω is an arbitrary open set such that $\overline{\omega}\subseteq R^n\times R^n-\Delta$.*

(3°) *$\lambda^q E(x, \xi\,|\,\lambda)$ belongs to $\mathscr{D}^{-[n/2]-1+2m}_{xL^2}$ and remain bounded in $\mathscr{D}^{-[n/2]-1+2m-2mq}_{xL^2}$ with respect to $(\xi, \lambda)\in R^n\times\Gamma$.*

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*) We denote $L(x, \xi\,|\,\lambda)$, $\Phi_\delta(x, \xi\,|\,\lambda)$, $L'(x, \xi\,|\,\lambda)$ and $\Phi'_\delta(x, \xi\,|\,\lambda)$ as follows:

$$\left\{A\left(x, \frac{\partial}{\partial x}\right)+\lambda\right\}E(x, \xi\,|\,\lambda)=\delta_{x-\xi}-L(x, \xi\,|\,\lambda),$$

$$\Phi_\delta(x, \xi\,|\,\lambda)=\left\{A\left(x, \frac{\partial}{\partial x}\right)+\lambda\right\}[\{1-\alpha_\delta(x-\xi)\}E(x, \xi\,|\,\lambda)], \text{ etc.}$$

(4°) *Let k be any non-negative integer. Then the linear mapping*: $\mathcal{D}_\xi^{k+2[n/2]+2+2m(q-1)} \to \mathcal{E}_x^k$ *defined by* $\varphi(\xi) \to \lambda^q \langle E(x, \xi \mid \lambda), \varphi(\xi)\rangle_\xi$ *and the mapping*: $\mathcal{D}_x^{k+2[n/2]+2+2m(q-1)} \to \mathcal{E}_\xi^k$ *defined by* $\psi(x) \to \lambda^q \langle E(x, \xi \mid \lambda), \psi(x)\rangle_x$ *are equi-continuous in λ. And these mappings continuously depend on λ.*

We construct $E(x, \xi \mid \lambda)$ as follows (c.f. S. Mizohata [3]):

$$(3.2) \quad P\left(\xi \mid \lambda; \frac{\partial}{\partial x}\right) = \sum_{|\nu|=2m} a_\nu(\xi)\left(\frac{\partial}{\partial x}\right)^\nu + 1 + \lambda.$$

$$(3.3) \quad Q\left(x, \frac{\partial}{\partial x}; \xi\right) = \sum_{|\nu|=2m} \{a_\nu(\xi) - a_\nu(x+\xi)\}\left(\frac{\partial}{\partial x}\right)^\nu - \sum_{|\nu|<2m} a_\nu(x+\xi)\left(\frac{\partial}{\partial x}\right)^\nu + 1,$$

$$(3.4) \quad f_0(\xi \mid \lambda) = P\left(\xi \mid \lambda; \frac{\partial}{\partial x}\right)^{-1} \delta_x = \frac{1}{(2\pi)^n} \int \frac{e^{i\langle x, y\rangle}}{P(\xi \mid \lambda; iy)} dy$$

$$(3.5) \quad f_i(\xi \mid \lambda) = P\left(\xi \mid \lambda; \frac{\partial}{\partial x}\right)^{-1} Q\left(x, \frac{\partial}{\partial x}; \xi\right) f_{i-1}(\xi \mid \lambda), \quad i = 1, 2, \cdots, \text{ and}$$

$$(3.6) \quad E(x, \xi \mid \lambda) = \tau_\xi[f_0(\xi \mid \lambda) + \cdots + f_j(\xi \mid \lambda)]$$

where τ_ξ is the translation operator by ξ: $\tau_\xi[\varphi](x) = \varphi(x-\xi)$. It is easy to show that (2°), (3°) and (4°) are satisfied by this $E(x, \xi \mid \lambda)$ whatever j may be, and that (1°) is satisfied if $j \geqq 2[n/2] + 2mq + p + 1$.[**)]

THEOREM. *The operator $A^{-\alpha}$($\mathrm{Re}\,\alpha > 0$) has a kernel representation*

$$(3.7) \quad (A^{-\alpha} f)(\xi) = \int_D K^{(\alpha)}(\xi, x) f(x) dx, \text{ for } f(x) \in L^2(D)$$

where the kernel

$$(3.8) \quad K^{(\alpha)}(\xi, x) = \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{-\alpha} K(\xi, x \mid \lambda) d\lambda$$

is very regular (c.f. L. Schwartz [4]): more precisely

i) *$K^{(\alpha)}(\xi, x)$ is infinitely differentiable in $(\xi, x) \in D \times D - \Delta$.*

ii) *This kernel maps continuously $\mathcal{D}_x(D)$ into $\mathcal{E}_\xi(D)$ and $\mathcal{E}'_x(D)$ into $\mathcal{D}'_\xi(D)$ and moreover, if $f(x) \in \mathcal{E}'_x(D)$, $\langle K^{(\alpha)}(\xi, x), f(x)\rangle_x$ is infinitely differentiable in ξ where $f(x)$ is infinitely differentiable.*

Remark: The mapping: $\mathcal{D}_x^{k+2[n/2]+2}(D) \to \mathcal{E}_\xi^k(D)$ defined by

$$(3.9) \quad \psi(x) \to \langle K^{(\alpha)}(\xi, x), \psi(x)\rangle_x$$

and the mapping: $\mathcal{D}_\xi^{k+2[n/2]+2}(D) \to \mathcal{E}_x^k(D)$ defined by

$$(3.10) \quad \varphi(\xi) \to \langle K^{(\alpha)}(\xi, x), \varphi(\xi)\rangle_\xi$$

are continuous.

COROLLARY. For any complex number α (not necessarily $\mathrm{Re}\,\alpha > 0$), the integral (3.8) converges and it is an entire function of α outside of the diagonal Δ, moreover this kernel is very regular.

In the case where $\mathrm{Re}\,\alpha \leqq 0$, the mappings (3.9) and (3.10) define continuous linear mappings: $\mathcal{D}_x^{k+2[n/2]+2+2mq'}(D) \to \mathcal{E}_\xi^k(D)$ and $\mathcal{D}_\xi^{k+2[n/2]+2+2mq'}(D) \to \mathcal{E}_x^k(D)$ respectively, where $q' = -[\mathrm{Re}\,\alpha] + 1$.

**) We can similarly construct a family of parametrix $E'(x, \xi \mid \lambda)$ satisfying this proposition.

References

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