

127. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. IX

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(Comm. by Kinjirō KUNUGI, M.J.A., Oct. 12, 1963)

In this paper we shall discuss, under some conditions, the relation between the distribution of ζ -points of the function $S(\lambda)$ defined in the statement of Theorem 1 [cf. Proc. Japan Acad., Vol. 38, 263–268 (1962)] and that of ζ -points of the ordinary part $R(\lambda)$ of $S(\lambda)$, on the supposition that $R(\lambda)$ is a polynomial in λ of degree less than or equal to d .

Theorem 23. Let $S(\lambda)$ and $\{\lambda_\nu\}$ be the same notations as those defined in the statement of Theorem 1; let the ordinary part $R(\lambda)$ of $S(\lambda)$ be a polynomial in λ of degree less than or equal to d ; let α be one of ζ -points of $R(\lambda)$ for an arbitrarily given complex number ζ ; let ρ and μ be arbitrarily prescribed positive numbers satisfying the conditions $\sup|\lambda_\nu| < \rho < \infty$ and $0 < \mu < 1$ respectively; let r be a positive number such that $\frac{\rho}{\mu} \leq r < \infty$; let $m_\kappa(r, \alpha)$ denote the minimum of the modulus $|R(\lambda) - \zeta|$ on the circle $|\lambda - \alpha| = r$; and let $K = \frac{M_s(\rho, 0)}{(1 - \mu)\rho^d}$ where $M_s(\rho, 0)$ denotes the maximum of the modulus $|S(\lambda)|$ on the circle $|\lambda| = \rho$. If

$$|\alpha| > \frac{Kr^{d+1}}{m_\kappa(r, \alpha)} + 2r,$$

then, in the interior of the circle $|\lambda - \alpha| = r$, $S(\lambda)$ has ζ -points whose number (counted according to multiplicity) is equal to that of ζ -points of $R(\lambda)$ in the interior of the same circle.

Proof. Since, by hypotheses, $R(\lambda)$ is a polynomial in λ of degree less than or equal to d ,

$$M_s\left(\frac{\rho}{\kappa}, 0\right) \leq \frac{M_s(\rho, 0)}{(1 - \mu)\kappa^d} \quad (0 < \kappa \leq \mu),$$

as we have already shown in the course of the proof of Theorem 13 stated in Part V [1]. Substituting $\frac{M_s(\rho, 0)}{1 - \mu}$ in this inequality by $K\rho^d$, we have therefore

$$(23) \quad M_s(r, 0) \leq Kr^d \left(\frac{\rho}{\mu} \leq r < \infty \right).$$

Moreover, if we denote by $\chi(\lambda)$ the sum of the first and the second principal parts of $S(\lambda)$,

$$\left| \chi\left(\frac{\rho}{\kappa} e^{i\theta}\right) \right| \leq \frac{\kappa}{1-\kappa} M_s(\rho, 0) \quad (0 < \kappa < 1) \quad [2],$$

as can be verified from either of the extended Fourier series expansion and the integral expression of $\chi(\lambda)$ in the domain $\{\lambda: |\lambda| > \rho\}$; and putting $\kappa(\theta) = \frac{r}{|\alpha + re^{i\theta}|}$, it is found from this result that the inequality

$$|\chi(\alpha + re^{i\theta})| \leq \frac{\kappa(\theta)}{1-\kappa(\theta)} M_s(r, 0) \quad \left(\frac{\rho}{\mu} \leq r < \infty\right)$$

is valid, as far as $|\alpha| > 2r$. Since $\frac{\kappa(\theta)}{1-\kappa(\theta)}$ is a monotone-increasing function of $\kappa(\theta)$ with $\frac{r}{|\alpha|+r} \leq \kappa(\theta) \leq \frac{r}{|\alpha|-r} < 1$ under the supposition that $|\alpha| > 2r$, we have therefore

$$|\chi(\alpha + re^{i\theta})| \leq \frac{r}{|\alpha|-2r} M_s(r, 0).$$

Thus it follows from the just established inequality and (23) that

$$|\chi(\alpha + re^{i\theta})| \leq \frac{Kr^{d+1}}{|\alpha|-2r} \quad \left(\frac{\rho}{\mu} \leq r < \infty\right),$$

as far as $|\alpha| > 2r$. In addition, if α satisfies the given condition

$$|\alpha| > \frac{Kr^{d+1}}{m_R(r, \alpha)} + 2r,$$

then there is no difficulty in showing from the last result that

$$|R(\alpha + re^{i\theta}) - \zeta| > |\chi(\alpha + re^{i\theta})| \quad \left(\frac{\rho}{\mu} \leq r < \infty\right).$$

On the other hand, since $|\alpha + re^{i\theta}| > r$ and $\sup_v |\lambda_v| < \frac{\rho}{\mu} \leq r < \infty$, it is obvious that $\chi(\lambda)$ is regular inside and on the circle $|\lambda - \alpha| = r$, and of course, so also is $R(\lambda) - \zeta$. In consequence, in the interior of the circle $|\lambda - \alpha| = r$ the function $S(\lambda) - \zeta = \{R(\lambda) - \zeta\} + \chi(\lambda)$ has zero-points whose number (counted according to multiplicity) is equal to that of zero-points of $R(\lambda) - \zeta$ in the interior of the same circle as it, according to the Rouché theorem.

The theorem has thus been proved.

Theorem 24. Let $S(\lambda)$, $R(\lambda)$, $\{\lambda_v\}$, d , ζ , ρ , μ , r , $M_s(\rho, 0)$, and K be the same notations as before; let α be one of ζ -points of $S(\lambda)$; and let $m_R(r, \alpha)$ denote the minimum of the modulus $|R(\lambda) - \zeta|$ on the circle $|\lambda - \alpha| = r$. If

$$|\alpha| > \frac{2Kr^{d+1}}{m_R(r, \alpha)} + 2r,$$

then, in the interior of the circle $|\lambda - \alpha| = r$, $R(\lambda)$ has ζ -points whose number (counted according to multiplicity) is equal to that of ζ -points of $S(\lambda)$ in the interior of the same circle as it.

Proof. If α satisfies the given condition concerning its modulus, then

$$\frac{|R(\alpha + re^{i\theta}) - \zeta|}{|\chi(\alpha + re^{i\theta})|} > 2,$$

as can be found by following the argument used in the proof of Theorem 23, and hence

$$|S(\alpha + re^{i\theta}) - \zeta| > |\chi(\alpha + re^{i\theta})|,$$

where both $S(\lambda) - \zeta$ and $\chi(\lambda)$ are regular inside and on the circle $|\lambda - \alpha| = r$. This final inequality implies that in the interior of the circle $|\lambda - \alpha| = r$ the function $R(\lambda) - \zeta = \{S(\lambda) - \zeta\} - \chi(\lambda)$ has zero-points whose number (counted according to multiplicity) is equal to that of zero-points of $S(\lambda) - \zeta$ in the interior of the same circle as it, according to the rewritten Rouché theorem quoted to prove Theorem 17 in Part VI.

With this result the present theorem has been proved.

References

- [1] S. Inoue: Some applications of the functional-representations of normal operators in Hilbert spaces. V, Proc. Japan Acad., **38**, 706-710 (1962).
 [2] —: Ibid., 706-708.

Correction to Sakuji Inoue: "Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. V" (Proc. Japan Acad., **38**, 706-710 (1962)).

Page 706, line 14: For " $M_S(r, 0) \geq Kr^d$ " read " $M_S(r, 0) \leq Kr^d$ ".

Corrections to S. Inoue: "On the Functional-Representations of Normal Operators in Hilbert Spaces" (Proc. Japan Acad., **38**, 18-22 (1962)), and "Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces" (Proc. Japan Acad., **38**, 263-268 (1962)).

Page 18, line 4 from the foot, and Page 263, line 9: Add "If the whole subset with non-zero measure of the continuous spectrum of N lies on a circumference with center at the origin," in front of "then".