125. On the Product of Paracompact Spaces

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1. Introduction. As is well known, the topological product of two paracompact Hausdorff spaces is not normal in general. The cases for which the topological product $X\times Y$ of a Hausdorff space X with any paracompact Hausdorff space Y has been proved to be normal are as follows:

(a) X is compact (J. Dieudonné [1]).

(b) X is σ-compact and regular (E. Michael [3]).

(c) X is paracompact and locally compact (K. Morita [7]).

In this paper we shall show that these cases can be unified into a single case. Namely, we shall establish the following theorem

Theorem 1. *Let X be a paracompact normal space which is a countable union of locally compact closed subsets, and let Y be a paracompact normal space. Then the product space $X\times Y$ is paracompact and normal.*[1)]

As an example of a paracompact normal space which is a countable union of locally compact closed subsets we can mention a CW-complex in the sense of J. H. C. Whitehead [16]. It is known (cf. C. H. Dowker [2, p. 563]) that the topological product of two CW-complexes is a closure finite cell complex but not a CW-complex in general. Theorem 1 shows that not only the product of two CW-complexes but also the product of a CW-complex with any paracompact normal space is paracompact and normal.

In Theorem 1 the condition that X be a countable union of locally compact closed subsets cannot be weakened further, at least so long as X is an M-space. Indeed, we have the following theorem, which gives a partial answer to a problem raised by H. Tamano [15].

Theorem 2. *Let X be an M-space, or more generally, a countable union of closed subsets each of which is an M-space. Then in order that the product space $X\times Y$ be normal for any paracompact normal space Y it is necessary and sufficient that X be a paracompact normal space which is a countable union of locally compact closed subsets.*

The notion of M-spaces was introduced and discussed in our previous paper [11]. Countably compact spaces, metrizable spaces, and

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1) It should be noted that the Hausdorff or T_1 separation axiom is not assumed for paracompact normal spaces throughout this paper.

paracompact Hausdorff spaces which are complete in the sense of E. Čech are M-spaces. CW-complexes are not M-spaces in general. Any paracompact normal space which is a countable union of locally compact closed subsets is a countable union of closed M-subspaces.

The proof of Theorem 2 rests upon recent results of E. Michael [5] and A. H. Stone [14]. The problem whether Theorem 2 is true without any restriction on X remains open.

In connection with Theorem 1 we obtain the following theorem concerning the covering dimension of product spaces.

Theorem 3. *Under the same assumptions for X and Y as in Theorem 1 we have*

$$\dim(X\times Y)\leqq \dim X+\dim Y.$$

In particular, if X is a CW-complex, then the equality holds.

2. A lemma. Lemma 1. *The following statements are equivalent for a paracompact normal space X.*[2]

(a) *X is a countable union of locally compact closed subsets.*

(b) *X is a union of a σ-locally finite system of compact closed subsets.*

Furthermore, if X is perfectly normal and Hausdorff (a) *is equivalent to* (a′) *and if X is regular* (b) *is equivalent to* (b′):

(a′) *X is a countable union of locally compact subsets.*

(b′) *X is a union of a σ-locally finite system of compact subsets.*

Proof. We shall prove only (a′)→(a); the other implications are obvious or easy to prove. Suppose that X is a countable union of locally compact subspaces A_i, $i=1,2,\cdots$. Since A_i is locally compact, A_i is the intersection of an open subset and a closed subset. Since X is perfectly normal, A_i is an F_σ. Let A_{ij}, $j=1,2,\cdots$ be closed subsets of X such that $A_i=\cup\{A_{ij}\mid j=1,2,\cdots\}$. Then X is the union of locally compact closed subsets A_{ij}, $i=1,2,\cdots$, $j=1,2,\cdots$.

3. Proof of Theorem 1. Let X be a paracompact normal space which is a countable union of locally compact closed subsets and Y a paracompact normal space. Then by Lemma 1 there exists a σ-locally finite closed covering $\{A_{i\alpha}\mid \alpha\in\Omega_i,\ i=1,2\cdots\}$ of X such that $\{A_{i\alpha}\mid\alpha\in\Omega_i\}$ is locally finite in X for each i and each subset $A_{i\alpha}$ is compact.

Since X is paracompact and normal, by [6, §3, Lemma] there are open subsets $L_{i\alpha}$, $\alpha\in\Omega_i$, $i=1,2\cdots$, such that

(1) $A_{i\alpha}\subset L_{i\alpha}$.

(2) $\{L_{i\alpha}\mid\alpha\in\Omega_i\}$ is locally finite in X for each i.

Let $\mathfrak{M}$ be any open covering of $X\times Y$. Then, for each point y of Y and for each $A_{i\alpha}$, we can find an open neighborhood $V(y)$ of y in Y and a finite system $\{U_j\mid j=1,\cdots,s\}$ of open subsets of X such

2) Cf. Footnote 1).

that

(3) $$U_j \times V(y) \subset \text{some set of } \mathfrak{M} \ (j=1,\cdots,s);$$

(4) $$A_{i\alpha} \subset \bigcup_{j=1}^{s} U_j \subset L_{i\alpha}.$$

This is easily verified since $A_{i\alpha}$ is compact. From (4) and the normality of $A_{i\alpha}$ it follows that we can find closed subsets F_j, $j=1,\cdots,s$, of X such that

(5) $$A_{i\alpha} = \bigcup_{j=1}^{s} F_j;\ F_j \subset U_j,\ j=1,2,\cdots,s.$$

Since X is normal, there exists open F_σ subsets U'_j, $j=1,\cdots,s$, such that $F_j \subset U'_j \subset U_j$.

If we let y range over all the points of Y, the family of all such $V(y)$ forms an open covering of Y. Since Y is paracompact and normal, this covering is refined by a locally finite covering of Y which consists of open F_σ subsets.

Thus for each $A_{i\alpha}$ we can find a locally finite covering

$$\mathfrak{H}(i,\alpha)=\{H(\lambda;i,\alpha)\,|\,\lambda\in\Lambda(i,\alpha)\}$$

of Y by open F_σ subsets and a family of finite systems

$$\mathfrak{G}(\lambda;i,\alpha),\quad \lambda\in\Lambda(i,\alpha)$$

consisting of open F_σ subsets of X such that

(6) $$A_{i\alpha}\subset \cup\{G\,|\,G\in\mathfrak{G}(\lambda;i,\alpha)\}\subset L_{i\alpha}\quad \text{for } \lambda\in\Lambda(i,\alpha),$$

(7) $$G\times H(\lambda;i,\alpha)\subset \text{some set of } \mathfrak{M} \text{ for } G\in\mathfrak{G}(\lambda;i,\alpha).$$

Let us put

(8) $$\mathfrak{K}(i,\alpha)=\{G\times H(\lambda;i,\alpha)\,|\,G\in\mathfrak{G}(\lambda;i,\alpha);\ \lambda\in\Lambda(i,\alpha)\}$$

(9) $$\mathfrak{K}_i=\cup\{\mathfrak{K}(i,\alpha)\,|\,\alpha\in\Omega_i\}$$

(10) $$\mathfrak{K}=\cup\{\mathfrak{K}_i\,|\,i=1,2,\cdots\}.$$

Then the union of all the sets in $\mathfrak{K}(i,\alpha)$ contains $A_{i\alpha}\times Y$. Hence the family $\mathfrak{K}$ is an open covering of $X\times Y$. From the construction it is obvious that $\mathfrak{K}$ is a refinement of $\mathfrak{M}$.

We shall prove that $\mathfrak{K}_i$ is locally finite in $X\times Y$ for each i. For this purpose, let (x_0, y_0) be any point of $X\times Y$. Then there exists an open neighborhood U_0 of x_0 in X such that U_0 intersects only finitely many elements of $\{L_{i\alpha}\,|\,\alpha\in\Omega_i\}$. Let $\Gamma_0=\{\alpha\in\Omega_i\,|\,L_{i\alpha}\frown U_0\neq\phi\}$; then Γ_0 is a finite set and we have

$$(U_0\times Y)\frown K=0\quad \text{for}\quad K\in\mathfrak{K}(i,\alpha) \text{ with } \alpha\notin\Gamma_0.$$

For each $\alpha\in\Gamma_0$ we can find an open neighborhood V_α of y_0 such that V_α intersects only finitely many elements of $\mathfrak{H}(i,\alpha)$. We put

$$V_0=\frown\{V_\alpha\,|\,\alpha\in\Gamma_0\}.$$

Since Γ_0 is a finite set, V_0 is an open neighborhood of y_0 in Y.

Now it is easy to see that $U_0\times V_0$ intersects only finitely many elements of $\mathfrak{K}_i$. Thus $\mathfrak{K}_i$ is locally finite.

On the other hand, each set of $\mathfrak{G}(\lambda;i,\alpha)$ is an open F_σ set in X and each set of $\mathfrak{H}(i,\alpha)$ is an open F_σ set in Y. Hence for each set

$G\times H(\lambda; i, \alpha)$ of $\mathfrak{R}(i, \alpha)$ there exists a non-negative continuous function φ over $X\times Y$ such that

$$G\times H(\lambda; i, \alpha)=\{(x, y)\mid \varphi(x, y)>0\};$$

we have only to put $\varphi(x, y)=f(x)g(y)$ for $x\in X$, $y\in Y$ where $f: X\rightarrow I$ and $g: Y\rightarrow I$ ($I=[0, 1]$) are continuous maps such that $G=\{x\mid f(x)>0\}$, $H(\lambda; i, \alpha)=\{y\mid g(y)>0\}$. Therefore by [12, Theorem 1.2] we see that $\mathfrak{R}$ is a normal covering of $X\times Y$.

Since $\mathfrak{R}$ is a refinement of $\mathfrak{M}$, $\mathfrak{M}$ is a normal covering of $X\times Y$. This shows that $X\times Y$ is paracompact and normal. Thus the proof of Theorem 1 is completed.

Corollary to Theorem 1. *Let X and Y be as in Theorem 1. In addition, if X is a perfectly normal space which is a countable union of closed metrizable subspaces, and if Y is perfectly normal, then the product space $X\times Y$ is perfectly normal.*[3)]

Proof. Let $\{A_i\mid i=1, 2, \cdots\}$ be a closed covering of X such that each A_i is metrizable. Then $A_i\times Y$ is perfectly normal by Morita [11, Theorem 4.1] or Michael [3, Proposition 5]. Hence every open subset of $X\times Y$ is an F_σ since every open subset of $A_i\times Y$ is an F_σ subset of $X\times Y$. This proves that $X\times Y$ is perfectly normal since $X\times Y$ is normal by Theorem 1.

As is easily seen, every CW-complex has the same property as X in Corollary to Theorem 1.

4. Proof of Theorem 2. We have only to prove the necessity of the condition.

Let X be a topological space such that the product space $X\times Y$ is normal for any paracompact Hausdorff space Y. Then X is paracompact and normal by [9, Theorem 2.4]. Suppose that X is an M-space. Then by [11, Theorem 5.3] there exists a closed continuous map f from X onto a metric space T such that $f^{-1}(t)$ is compact for each point t of T; it is to be noted here that if a countably compact space is paracompact normal then it is compact. Let Y be any paracompact Hausdorff space. If we put

$$\varphi(x, y)=(f(x), y) \quad \text{for} \quad x\in X,\ y\in Y,$$

then φ is a closed continuous map from $X\times Y$ onto $T\times Y$ by [10, Lemma 2.1]. By assumption $X\times Y$ is normal and hence $T\times Y$ is normal for any paracompact Hausdorff space Y. Since T is metrizable it follows from a recent result of E. Michael [5] that T must be an absolute F_σ space for metric spaces, and hence by A. H. Stone [14] T must be a countable union of locally compact subsets. Hence by Lemma 1 there are a countable number of locally compact closed subsets B_i, $i=1, 2, \cdots$, such that $T=\cup B_i$. If we put $A_i=f^{-1}(B_i)$,

3) Cf. Morita [13, Theorem 3.2].

$i=1, 2, \cdots$, then $X=\cup A_i$, and A_i are closed in X. Since $f|A_i$ is a closed continuous map from A_i onto B_i such $(f|A_i)^{-1}(t)$ is compact for each point t of B_i, we see that A_i is locally compact. Thus X is a countable union of locally compact closed subsets.

Finally, let X be a countable union of closed subsets A_i, $i=1, 2, \cdots$, such that each A_i is an M-space. Since $A_i \times Y$ is normal for any paracompact Hausdorff space Y, A_i is a countable union of locally compact closed subsets as has been proved above. Therefore X is itself a countable union of locally compact closed subsets. This completes the proof for the necessity of the condition.

5. **Proof of Theorem 3.** Let X be a paracompact normal space which is a countable union of locally compact closed subsets A_i, $i=1, 2, \cdots$, and let Y be a paracompact normal space. Then we have

$$\dim (A_i \times Y) \leqq \dim A_i + \dim Y \leqq \dim X + \dim Y$$

by [7, Theorem 4] and hence by virtue of the sum theorem we have $\dim (X \times Y) \leqq \dim X + \dim Y$.

In particular, if X is a CW-complex K of dimension m, then K contains a closed subset which is homeomorphic to a closed m-simplex and hence we have $\dim (K \times Y) \geqq m + \dim Y$ by [7, Theorem 7]. This completes the proof of Theorem 3.

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