138. *The Relativity Theory in the Einstein Space under the Extended Lorentz Transformation Group*

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The general theory of relativity of A. Einstein was based on the non-definite quadratic differential form

(1) $$dS^2 = g_{\mu\nu}(x^\sigma)\,dx^\mu dx^\nu, \quad (\lambda, \mu, \nu, \cdots = 1, 2, 3, 4)$$

and grasped as the *Riemannian geometry of the Einstein space*:

(i) $R_{\mu\nu}=0$, | (ii) $R_{\mu\nu}=\frac{R}{4}g_{\mu\nu}$,

the path of a free particle being the geodesic curve:

(2) $$\frac{d^2x^\lambda}{dS^2}+\left\{{\lambda \atop \mu\nu}\right\}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}=0.$$

The fundamental assumption was the so-called *principle of equivalence.* The merit was the *geometrization of physics.* But the demerit was the obscurity of the physical side caused by the laborious calculations in terms of $g_{\mu\nu}$ and $\left\{{\lambda \atop \mu\nu}\right\}$ as well as by too much forcing physical interpretations. Thus the Einstein's theory has remained merely as a *conjecture* for the last 47 years without becoming a decisive immortal theory.

With the hope to make it a decisive theory comparable with the Newton's theory, the present author ([1]–[14]) started with the expressibility of (1) in the form

(3) $$dS^2 = g_{\mu\nu}dx^\mu dx^\nu = (-1)^{1+\delta_l^4}\omega^l\omega^l, \quad (\omega^l = \omega^l_\mu(x^\sigma)dx^\mu, |\omega^l_\mu| \neq 0)$$

except undergoing *extended* orthogonal transformations of $\frac{1}{2}(1+\delta_l^4)\omega^l$, having discovered the *extended* orthogonal transformations with functions of coordinates (x^σ) as coefficients and simplified calculations extremely by taking $\omega^l_\mu(x^\sigma)$ and $\Lambda^\lambda_{\mu\nu}$ in place of $g_{\mu\nu}=\omega^l_\mu\omega^l_\nu$ and $\{{\lambda \atop \mu\nu}\}$ respectively, where

(4) $$\Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \Omega^\lambda_l \frac{\partial\omega^l_\mu}{\partial x^\nu} \equiv -\omega^l_\mu \frac{\partial\Omega^\lambda_l}{\partial x^\nu}$$

is the *parameter of teleparallelism* of $\omega^l_\mu(x^\sigma)$ and $\Omega^\lambda_l(x^\sigma)$, and

(5) $$\Omega^\lambda_l\omega^l_\mu = \delta^\lambda_\mu \iff \Omega^\lambda_m\omega^l_\lambda = \delta^l_m,$$

the δ's being the Kronecker deltas. The equations of motion of a free particle were

(6) $$\frac{d^2\xi^l}{dS^2} = \frac{d}{dS}\frac{\omega^l}{dS} \equiv \omega^l_\lambda\left\{\frac{d^2x^\lambda}{dS^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}\right\}=0,$$

whose finite equations are

(7) $$\xi^l = a^l S + c^l, \quad (a^l, c^l\text{: const.}),$$
which represent the author's II-*geodesics* in 4 dimension, *which behave as for meet and join as well as for the extremal* $\delta S=0$ *like straight lines*, the identity (6) having been discovered by the present author. The (ξ^l) were called by the present author the II-*geodesic rectangular coordinates* referred to the II-*geodesic* ξ^l-*axes*. The (x^σ) might have been local curvelinear coordinates. But the author started with the Cartesian coordinates, etc.:

(8) $$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ir = ict, \quad (t = \text{time})$$
in order to make the physical side clear and transparent. He grasped ([9]-[10]) *his theory of general relativity* as his 3-*dimensional extended equiform Laguerre geometry under his extended equiform Laguerre transformation group of*

(9) $$\varepsilon_l \bar{\xi}^l = a^l_m(\xi^p)\varepsilon_m \xi^m + \varepsilon_l a^l_0, \quad (a^l_0 = \text{const.}, \ \varepsilon_l = \tfrac{1}{2}(1+\delta^4_l)),$$

(10) $$\varepsilon_l \xi^l = \omega^l_\mu(x^\sigma)\varepsilon_\mu x^\mu + \varepsilon_l \omega^l_0, \quad (\omega^l_0 = \text{const.}, \ \varepsilon_\mu = \tfrac{1}{2}(1+\delta^4_\mu)),$$
where $(a^l_m(\xi^p))$ and $(\omega^l_\mu(x^\sigma))$ are *orthogonal matrices with determinant* $\neq 0$. *The transformations* (9) *and* (10) (*accompanied by* (8)) *are extended Lorentz transformations* so-to-speak. The space element is an oriented sphere with center (x, y, z) and radius r or its maps by (9) including (10). The ds such that

(11) $$-ds^2 = (-1)^{1+\delta^4_l} dx^l dx^l > 0$$
is the (usually pure imaginary) common tangential segment of two consecutive oriented spheres (x^σ), $(x^\sigma + dx^\sigma)$. We utilize dS such that

(12) $$dS^2 = -ds^2 > 0,$$
and identify $\omega^l_\mu(x^\sigma)$ with the *momentum-potential vector*, so that dS is the *action* and S the *action function*. The II-geodesics (6) in 4 dimension are in 3 dimension "Kanalflächen" enveloped by *oriented* II-*geodesic spheres* with the particle (x^1, x^2, x^3) as center and a II-*geodesic radius* $\int \frac{\omega^4}{dS} dS$.

In this note, it will first be shown that *the relation*

(13) $$\frac{d^2\xi^l}{dS^2} = \frac{d}{dS}\frac{\omega^l}{dS} \equiv \omega^l_\lambda \left(\frac{d^2x^\lambda}{dS^2} + \Lambda^\lambda_{\mu\nu} \frac{dx^\mu}{dS}\frac{dx^\nu}{dS} \right) \equiv \omega^l_\lambda \left(\frac{d^2x^\lambda}{dS^2} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{dS}\frac{dx^\nu}{dS} \right)$$
holds and then we will *compare the two theories of relativity* of A. Einstein and the present author, so that the decisive eternity (comparable with that of Newton's law) of the present author's theory will become clear, while the Einstein's theory remains, contrary to our hope, merely as an historical conjecture. *The essential difference consists in the ways of identifications of the geometric objects with the physical objects and in the present author's 3-dimensional extended equiform Laguerre geometrical grasping of the geometrical law.*

First proof of (13). In the theory of an-holonomic system, the following relations are known:

$$(14)\qquad dS^2=g_{\mu\nu}dx^\mu dx^\nu=g_{hk}\omega^h\omega^k=g_{hk}\omega^h_\mu\omega^k_\nu dx^\mu dx^\nu,$$

$$(15)\qquad \frac{d}{dS}\frac{\omega^l}{dS}+\begin{Bmatrix} l \\ hk\end{Bmatrix}\frac{\omega^h}{dS}\frac{\omega^k}{dS}=\omega^l_\lambda\Big(\frac{d^2x^\lambda}{dS^2}+\begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}\Big),$$

where $\begin{Bmatrix} l \\ hk\end{Bmatrix}$ is constructed with g_{hk}. In case (3), we have

$$(16)\qquad g_{hk}=(-1)^{1+\delta^4_h}\delta_{hk},$$

so that $\begin{Bmatrix} l \\ hk\end{Bmatrix}=0$ and thus (15) becomes

$$(17)\qquad \frac{d^2\xi^l}{dS^2}=\omega^l_\lambda\Big(\frac{d^2x^\lambda}{dS^2}+\begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}\Big),$$

which, taken together with the author's identity (6), shows (13).

Second proof of (13). We know

$$(18)\qquad g_{\mu\nu}=\omega^l_\mu\omega^l_\nu,\quad g^{\mu\nu}=\Omega^\mu_l\Omega^\nu_l.$$

Hence

$$\begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}=\frac{1}{2}g^{\lambda\sigma}\Big(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\sigma\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\Big)$$

$$=-\frac{1}{2}\Omega^\lambda_h\Omega^\sigma_h\Big(\frac{\partial\omega^l_\mu}{\partial x^\nu}\omega^l_\sigma+\omega^l_\mu\frac{\partial\omega^l_\sigma}{\partial x^\mu}+\frac{\partial\omega^l_\sigma}{\partial x^\mu}\omega^l_\nu+\omega^l_\sigma\frac{\partial\omega^l_\nu}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\omega^l_\nu-\omega^l_\mu\frac{\partial\omega^l_\nu}{\partial x^\sigma}\Big),$$

$$(19)\qquad \begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}=\frac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\frac{1}{2}\Omega^\lambda_h\Omega^\sigma_h\Big\{\omega^l_\mu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\Big)+\omega^l_\nu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\Big)\Big\}.$$

We can show

$$(20)\qquad \Omega^\sigma_h\Big\{\omega^l_\mu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\Big)+\omega^l_\nu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\Big)\Big\}dx^\mu dx^\nu\equiv 0$$

as follows.

The left-hand side $=2\Omega^\sigma_h\omega^l_\mu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\Big)dx^\mu dx^\nu$

$$=2\omega^l\Big(\Omega^\sigma_h d\omega^l_\sigma-\frac{\partial\omega^l_\nu}{\omega^h}\Omega^\nu_p\omega^p\Big)=2\omega^l\Big(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_p\frac{\partial\omega^l_\nu}{\omega^q}\frac{\omega^q}{\omega^h}\omega^p\Big)$$

$$=2\omega^l(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_p d\omega^l_\nu\delta^p_h)=2\omega^l(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_h d\omega^l_\nu)=0.$$

Third proof of (13). According to [16], we set

$$(21)\qquad \begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}=\frac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu,$$

so that

$$(22)\qquad \delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu=\Omega^\lambda_h\Omega^\sigma_h\Big\{\omega^l_\mu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\Big)+\omega^l_\nu\Big(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\Big)\Big\}.$$

The contraction $\mu\rightarrow\lambda$ yields us

$$(23)\qquad (n+1)\psi_\nu=\Lambda^\sigma_{\sigma\nu}-\Lambda^\sigma_{\nu\sigma},$$

what shows us the relation (13).

Fourth proof of (13). We obtain

$$(24)\qquad \frac{d^2x^\lambda}{dS^2}+\begin{Bmatrix} \lambda \\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}=0 \quad\Big|\quad \frac{d^2x^\lambda}{dS^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}=0$$

as solutions of one and the same extremal problem $\delta S=0$, the variations of parameters being

$\delta x^\sigma,\ \delta\frac{dx^\sigma}{dS}.$	$\delta\frac{d\xi^l}{dS}=\frac{\partial\omega^l_\mu}{\partial x^\nu}\delta\frac{dx^\nu}{dS}\frac{dx^\mu}{dS}+\omega^l_\mu\delta\frac{dx^\mu}{dS}.$ (The cyclic case!)

The Meaning of the Relation (13).

The straight lines in the 4-dimensional Minkowski space are geodesic curves as well as II-geodesic curves at the same time. *The* II-*geodesic curves* $\frac{d^2\xi^l}{dS^2}=0$, $(\xi^l=a^lS+c^l)$ *are the maps of the straight lines* (24) *by the extended Lorentz transformation* (10), *the laws of meet, join and the extremal* $\delta S=0$ *being retained.*

Comparison of the Theories of Relativity of

A. Einstein.	T. Takasu.
1°. *Geometrization of physics.*	
2°. $0<dS^2=g_{\mu\nu}dx^\mu dx^\nu$.	$0<dS^2=(-1)^{1+\delta^4_l}\omega^l\omega^l$, $(\omega^l=\omega^l_\mu dx^\mu)$, *except undergoing extended equiform Laguerre transformations.*
3°. $g_{\mu\nu}(x^\sigma)$: generalized gravitational potential.	$\omega^l_\mu(x^\sigma)$: *momentum-potential vector 2 way components, gravitational or electromagnetic, or both.*
4°. One starts	One starts with $(x^\sigma)=(x, y, z, ir)$, ω^l *being written in invariant form,* and afterwards
with curvelinear coordinates (x^σ).	
5°. Interval dS.	*Action* dS.
6°. *Receptacle of physical phenomena:*	
space-time $\{x^\sigma\}\rightarrow$Einstein space.	Cartesian space (x, y, z), t being treated as in the classical manner.
7°. *Path of a free particle:* $\frac{d^2x^\lambda}{dS^2}+\left\{{\lambda \atop \mu\nu}\right\}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}=0$: *geodesic in the Einstein space.* Cf. (13).	$\frac{d^2\xi^l}{dS^2}=\omega^l_\lambda\left(\frac{d^2x^\lambda}{dS^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dS}\frac{dx^\nu}{dS}\right)=0$: II-*geodesic in* 4 *dimension=series of oriented* II-*geodesic spheres* $\left(x, y, z; \int\frac{\omega^4}{dS}dS\right)$.
8°. *Riemannian geometry of Eistein space.*	*Extend equiform Laguerre geometry.*
9°. Group of transformations $\bar{x}^\lambda=\bar{x}^\lambda(x^\sigma)$ preserving dS^2: $\left\lvert\frac{\partial\bar{x}^\lambda}{\partial x^\sigma}\right\rvert=0$.	3-*dimensional extended equiform Laguerre (extended Lorentz) transformation group* (9), (10).
10. Physical change.	*Extended equiform Laguerre transformation.*

11°. (i) Schwarzschild's form:

$$dS^2=\gamma(\rho)dt^2-\gamma(\rho)^{-1}d\rho^2-\rho^2d\theta^2-\rho^2\sin^2\theta d\varphi^2,\ \left(\gamma(\rho)=1-\frac{2m}{\rho}\right);$$

(ii) Takasu's form: $dS^2=\gamma(\rho)dt^2-\bar{\gamma}(\rho)^{-1}d\rho^2-\rho^2d\theta^2-\rho^2\sin^2\theta d\varphi^2,$

$$\left(-\bar{\gamma}=h^2\left(1-2mu-\frac{2m}{h^2u}\right)+\frac{C}{u^2},\ u=\frac{1}{\rho};\ h,\ C=\text{const.}\right).$$

(i)	(i), (ii)

$\rightarrow$planetary orbit: $$\frac{d^2u}{d\varphi^2}+u=\frac{m}{h^2}+3mu^2,$$

supported by 3 famous observations.

12°. *Principle of equivalence.*	*Invariancy of physical phenomena by extended equiform Laguerre transformations.*
13°. *Relativity.* $\leftarrow$	*Referring to moving coordinate system* (ξ^l).
14°. *Gravitation theory.*	*Physics of acceleration.*
15°. *Gravitational wave, Maxwell's equations (approximation theory).*	*Exact gravitational wave, exact Maxwell's equations* ([9], [10], [14]).
16°. * * *	*Schrödinger-Goldon equation referred to moving coordinate system* (ξ^l) [9].
17°. * * *	*Dirac equation referred to moving coordinate system* (ξ^l) [9].
18°. * * *	*Principle of least work:* $\delta\frac{dS}{dt}=0$ $\rightarrow$*equations of force lines* (II-*geodesic curves*).
19°. *Special relativity:* $dS^2=c^2dt^2-dx^2-dy^2-dz^2$, *under the Lorentz group, the space element being a point in the Minkowski space.*	*Physics of uniform motion:* $dS^2=(c^2dt)^2-(cdx)^2-(cdy)^2-(cdz)^2$, *under the Laguerre group, the space element being an oriented sphere with center* (x, y, z) *and radius* $r=c^2t$.

FitsGerald factor $\left(1-\frac{dx^2+dy^2+dz^2}{c^2dt^2}\right)^{-\frac{1}{2}}$

$=c\frac{dt}{dS}.$	$=c^2\frac{dt}{dS}.$

20°. *Classical physics, theory of special relativity, gravitation theory, electromagnetic theory, and the universally accepted part of the quantum theory are*

not unified.	*unified.*
21°. *An approximation theory, mere conjecture.*	*Decisive exact theory with eternity character as the Newton's theory.*

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