163. *A Characterization of Regular Rings*

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1963)

A ring A is said to be regular in the sense of von Neumann [4] if, for every element $a \in A$, there exists an element $x \in A$ such that $axa = a$. A subring M of an arbitrary ring A is called a quasiideal of A if $AM \cap MA \subseteq M$. In a recent paper [3], Lajos proved that in a regular ring a subring M of A is a quasiideal of A if and only if $MAM \subseteq M$. The question, whether or not this is a characteristic property of the regular rings, seems to be of some interest. In the present note we will give a theorem relative to this problem.

Theorem. For an arbitrary ring A, the following conditions are equivalent:

(i) A is regular.

(ii) For every subring M of A, $MAM \subseteq M$ implies $MAM = M$.

(iii) For every quasiideal M of A, $MAM = M$.

Proof. (i) implies (ii). Let M be a subring of A. If $MAM \subseteq M$ and if $a \in M$, then, by the regularity of A, $a = axa$ for some $x \in A$, so $a \in MAM$. Hence $M \subseteq MAM$ and so $M = MAM$.

(ii) implies (iii). Let M be a quasiideal of A, i.e. $MA \cap AM \subseteq M$. Since $MAM \subseteq MA \cap AM$, $MAM \subseteq M$, and hence by (ii) $MAM = M$.

(iii) implies (i). Let a be an element in A and let $M = (a)_R \cap (a)_L$ where $(a)_R$ and $(a)_L$ are, respectively, the right ideal and left ideal of A generated by a. It is easy to see that M is a quasiideal of A. Hence by (iii) $a \in M = MAM \subseteq (a)_R A (a)_L$. Thus, $a = axa$ for some $x \in A$, and A is regular.

Remark: A ring A having the property: "for any subring M of A, $MAM \subseteq M$ implies that M is a quasiideal of A" need not be regular. The ring of all rational integers is an example of such kind of rings.

The concept of regular semigroup and quasiideal of a semigroup have been defined analogously by Green [1] and Lajos [2] respectively. An analogous characterization theorem for regular semigroups can be similarly proved.

References

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