162. A Note on the Asymptotic Behaviour of a Power Series near its Circle of Convergence

By M. Abrar ALAM
Tarkapur, Mirzapur (U.P.), India
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1. Let $\sum a_n$ be an infinite series having partial sums

$$s_n = a_0 + a_1 + \cdots + a_n,$$

and let S_n^α and s_n^α denote respectively the nth Cesàro sum and mean of order α of the sequence $\{s_n\}$. Then

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu,$$

and

$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha},$$

where

$$\sum_{n=0}^{\infty} A_n^\alpha z^n = (1-z)^{-\alpha-1} \quad \text{for} \quad |z|<1,$$

so that

$$A_n^\alpha = \binom{n+\alpha}{n}$$

and, for $\alpha \neq -1, -2, \cdots,$

$$A_n^\alpha \sim n^\alpha / \Gamma(\alpha+1),$$

as $n \to \infty$.

A series $\sum a_n$ is said to be *summable* (C, α) to the sum s, if $\lim_{n\to\infty} s_n^\alpha = s$.

2. Let a complex power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

whose radius of convergence we take, for the sake of simplicity, to be unity, represent the function $f(z)$ within the circle $|z|<1$.

The function represented by the power series is always regular inside the circle. But no information, in general, can be deduced about the regularity of the function at a point of the circumference from the mere knowledge of the convergence or divergence of the series at the point.

However, in certain special cases, some results are known which relate the coefficients a_n with the analytic properties of the function $f(z)$ on the circle of convergence. Again, special methods such as Borel's method of summability (B), have been devised to associate a 'sum-function' of the power series outside the circle, and thus obtain an analytic continuation of the function.

The first theorem in this connection is the well known Abel's theorem which states that *if* $\sum a_n$ *converges to the sum* s, *then* $f(z)\to s$ *as* $z\to 1$, *along any path lying between two chords of the unit circle which pass through* $z=1$.[1] To obtain a converse of this theorem we have to impose certain restrictions upon the coefficients a_n.

When the series is divergent, we may seek to obtain the asymptotic behaviour of the function $f(z)$ as $z\to 1$; and, conversely, if $f(z)$ fails to tend to a finite limit as $z\to 1$, we may seek to obtain the asymptotic behaviour of the sequence of partial sums $\{s_n\}$ as $n\to\infty$.

If $\sum a_n$ is a series of non-negative terms we can obtain the asymptotic behaviour of the function $f(z)$ as $z\to 1$ by comparing the coefficients a_n of the power series $\sum a_n z^n$ with those of a known one, with the help of the following theorem.[2]

THEOREM A. *Let*

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\quad g(z)=\sum_{n=0}^{\infty}b_n z^n,$$

where $a_n\geqq 0, b_n\geqq 0$ *and the series converge for* $|z|<1$ *and diverge for* $z=1$. *If, as* $n\to\infty$,

$$a_n\sim Cb_n,$$

then, as $z\to 1$,[3] $f(z)\sim Cg(z)$.

A direct application of this theorem leads to the following

THEOREM B.[4] *If* $p>0$, *and* $s_n\sim Cn^p$, *then*

$$f(z)\sim C\frac{\Gamma(p+1)}{(1-z)^p}.$$

The converse of this theorem, stated below, has been given by Hardy and Littlewood[5] for the case in which the coefficients a_n are all non-negative.

THEOREM C. *If*

$$f(z)\sim\frac{C}{(1-z)^p},\quad a_n\geqq 0,\ p>0,$$

then

$$s_n\sim\frac{C}{\Gamma(p+1)}n^p.$$

Introducing the notion of summability, we obtain the following extensions of Theorems B and C.

THEOREM 1. *Let* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *for* $|z|<1$. *If* $s_n^\alpha\sim Cn^p$, $\alpha\neq -1, -2,\cdots$, *and* $\alpha+p>0$; *or* $S_n^\alpha\geq 0$ *for all* n *and* $\sum S_n^\alpha=\infty$, $\alpha+p>-1$, *then*

1) Titchmarsh [3], p. 229, §7.61.

2) Titchmarsh [3], p. 224, §7.5.

3) Here and hereafter $z\to 1$ means that z tends to 1 radially inside the circle of convergence $|z|<1$.

4) Hobson [2], p. 180.

5) Hardy and Littlewood [1], see also Hobson [2], p. 181.

$$f(z) \sim C \frac{\Gamma(\alpha+p+1)}{\Gamma(\alpha+1)} \frac{1}{(1-z)^p}$$

as $z \to 1$.

THEOREM 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$, *for* $|z|<1$. *If* $S_n^{\alpha-1} \geqq 0$ *for all values of* n, *and*

$$f(z) \sim \frac{C}{(1-z)^p}, \ \alpha+p>0, \ \alpha \neq -1, -2, \cdots,$$

as $z \to 1$, *then*

$$s_n^\alpha \sim C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+1)} n^p$$

as $n \to \infty$.

3. *Proof of Theorem* 1. Since $s_n^\alpha \sim Cn^p$, we have

(1) $$S_n^\alpha \sim C \frac{n^{\alpha+p}}{\Gamma(\alpha+1)}.$$

Now,

$$(1-z)^{-1} f(z) = \sum_{n=0}^{\infty} s_n z^n,$$

and hence

$$F(z) \equiv (1-z)^{-\alpha} f(z) = (1-z)^{-\alpha+1} \sum_{n=0}^{\infty} s_n z^n$$

(2) $$= (1-z) \left(\sum_{n=0}^{\infty} A_n^{\alpha-1} z^n \right) \left(\sum_{n=0}^{\infty} s_n z^n \right)$$

$$= (1-z) \sum_{n=0}^{\infty} S_n^\alpha z^n,$$

the multiplication being justified by the absolute convergence of the two series for $|z|<1$. In view of the relation (1) and the identity (2), we obtain, from Theorem B or A,

$$F(z) \sim C \frac{\Gamma(\alpha+p+1)}{\Gamma(\alpha+1)} \cdot \frac{1}{(1-z)^{\alpha+p}},$$

as $z \to 1$; whence the result follows.

Proof of Theorem 2. We have

$$f(z) \sim C(1-z)^{-p}.$$

Write

$$F(z) \equiv (1-z)^{-\alpha} f(z).$$

Then, by hypothesis

$$F(z) \sim \frac{C}{(1-z)^{\alpha+p}}.$$

Now

$$F(z) = (1-z)^{-\alpha+1} \sum_{n=0}^{\infty} s_n z^n$$

$$= \left(\sum_{n=0}^{\infty} A_n^{\alpha-2} z^n \right) \left(\sum_{n=0}^{\infty} s_n z^n \right)$$

$$= \sum_{n=0}^{\infty} S_n^{\alpha-1} z^n \qquad (|z|<1).$$

Hence, by Theorem C, we have

$$(S_0^{\alpha-1}+S_1^{\alpha-1}+\cdots+S_n^{\alpha-1})\sim\frac{C}{\Gamma(\alpha+p+1)}n^{\alpha+p},$$

or

$$S_n^\alpha\sim\frac{C}{\Gamma(\alpha+p+1)}n^{\alpha+p},$$

as $n\to\infty$; whence the result follows.

4. We give now a generalized form of Theorem A, viz.

THEOREM 3. *Let the series*

$$f(z)=\sum_{n=0}^\infty a_n z^n,\quad g(z)=\sum_{n=0}^\infty b_n z^n,$$

converge for $|z|<1$ *and diverge for* $z=1$, *and let* $S_n^\alpha\geqq 0$, $T_n^\beta\geqq 0$ *be the n-th Cesàro sums of orders* α *and* β *respectively. If* $\sum S_n^\alpha$ *and* $\sum T_n^\beta$ *diverge, and, as* $n\to\infty$,

$$S_n^\alpha\sim CT_n^\beta,$$

then, as $z\to 1$,

$$f(z)\sim C(1-z)^{\alpha-\beta}g(z).$$

Proof. Let us write

$$F(z)\equiv(1-z)^{-\alpha}f(z)=(1-z)\sum S_n^\alpha z^n$$

and
$$G(z)\equiv(1-z)^{-\beta}g(z)=(1-z)\sum T_n^\beta z^n.$$

Since
$$S_n^\alpha\sim CT_n^\beta,$$

we get, by Theorem A,

$$F(z)\sim CG(z),$$

or
$$(1-z)^{-\alpha}f(z)\sim C(1-z)^{-\beta}g(z),$$

whence the result follows.

A result of even more general form can be obtained by the introduction of the Nörlund method of summability, which includes the Cesàro method as a particular case.

Let $\{p_n\}$ be an infinite sequence, and let

$$P(z)=\sum_{n=0}^\infty p_n z^n$$

for $|z|<1$.

We write

$$N_n=\sum_{\nu=0}^n p_{n-\nu}s_\nu,$$

so that N_n is the nth Nörlund sum with the sequence of coefficients $\{p_n\}$, for the series $\sum a_n$.

We have, for $|z|<1$,

$$(1-z)^{-1}P(z)f(z)=\sum_{n=0}^\infty p_n z^n\sum_{n=0}^\infty s_n z^n$$
$$=\sum_{n=0}^\infty N_n z^n.$$

We can now easily verify

THEOREM 4. *Let the power series*

$$f(z)=\sum_{n=0}^\infty a_n z^n,\quad g(z)=\sum_{n=0}^\infty b_n z^n,$$

converge for $|z|<1$, *and let* $N_n \geqq 0$, $\tilde{N}_n \geqq 0$, *where* N_n *and* $\tilde{N}_n$ *denote the n-th Nörlund sums with the sequences* $\{p_n\}$ *and* $\{\tilde{p}_n\}$ *for the series* $\sum a_n$ *and* $\sum b_n$ *respectively. If* $\sum N_n$ *and* $\sum \tilde{N}_n$ *diverge and, as* $n \to \infty$,

$$N_n \sim C\tilde{N}_n,$$

then, as $z \to 1$,

$$f(z) \sim C\frac{\tilde{P}(z)}{P(z)}g(z),$$

where $P(z) \equiv \sum p_n z^n$ *and* $\tilde{P}(z) \equiv \sum \tilde{p}_n z^n$ *for* $|z|<1$.

Similar extensions can also be obtained for Theorems 1 and 2.

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References

[1] Hardy, G. H., and Littlewood, J. E.: Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, Proc. London Math. Soc. (2), **13**, 174–191 (1914).

[2] Hobson, E. W.: Theory of Functions of a Real Variable etc., Vol. II, Cambridge (1926).

[3] Titchmarsh, E. C.: The Theory of Functions, Oxford (1939).