5. On the Ambiguity of Cut-off Process in the Theory of Quantum Field

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1. In the calculations of quantum field theory, so called “cut-off” procedure is frequently used to remove divergence of some physical quantity. In this and the following [1] notes we discuss the ambiguities which is immanent in some of these cut-off procedures and consider some kind of remedies for it.

2. To clarify the cut-off procedures we consider the relation between the occupation number representation and the tensor product of Hilbert spaces in an axiomatic manner.

Assumptions with respect to the states of a single particle are the following:

$S_1$. The states of a single particle correspond to the vectors of a Hilbert space $H_1$.

$S_2$. The physical quantities of a single particle correspond to the self-adjoint operators (defined on a dense subset of the space $H_1$).

$S_3$. There is a set of physical quantities whose corresponding operators commute each other and makes a complete system of operators. We denote these operators by $O_1, \ldots, O_n$.

$S_4$. We restrict our considerations to the case that eigenvalues $\alpha_i (\alpha_1, \ldots, \alpha_n)$ of the operators $(O_1, \ldots, O_n)$ makes a point (discrete) spectrum.

Assumptions with respect to the states of quantized field are the following:

$F_1$. To every system of eigenvalues $\alpha_i (\alpha_{i_1}, \ldots, \alpha_{i_n})$ of single particle, i.e. to every eigenstate $\psi_i$ of the quantum number $\alpha_i$, one assigns the number $n_i$ of particles which are in this state. We assume that there exist eigenvectors $\varphi_{n_i}$ for every $n_i$ and a Hilbert space $H_i$ which is the closure of the linear aggregate of $\{\psi_{n_i} | n_i = 0, 1, 2, \ldots\}$.

$F_2$. Corresponding to a state $\Phi$ of the field, there exist vectors of the infinite direct product $H_1 \otimes H_i$ of the space $H_i$ in the sense of J. Von Neumann [2]. We leave details of the indicate direct product to the original author's article [2], and indicate only the following properties (1), (2) of $c$-sequences* which has close relation to the am-

* Notations and abbreviations in this article follow J. Von Neumann [2].
biguities of cut-off procedures.

(1) \[ \lambda_0 \Pi_{t=1}^\infty f_t = (\Pi_{t=1}^{m-1} f_t) \otimes \lambda_0 f_m \otimes (\Pi_{t=m+1}^\infty f_t) \]
for any integer \( m \) and for any complex number \( \lambda_0 \).

(2) \[ (\Pi_{t=1}^{m-1} f_t) \otimes (g_m + h_m) \otimes (\Pi_{t=m+1}^\infty f_t) \]
\[ = (\Pi_{t=1}^{m-1} f_t) \otimes g_m \otimes (\Pi_{t=m+1}^\infty f_t) + (\Pi_{t=1}^{m-1} f_t) \otimes h_m \otimes (\Pi_{t=m+1}^\infty f_t) \]
for any integer \( m \).

The vectors having the expressions of either side of these equalities hence define the same state.

3. We consider here some of usual cut-off process in the terms of tensor product. The cut-off procedure which appears in the customary calculation of quantum field theory is of the following kind.

With respect to some physical quantity \( \mathcal{M} \) which is expressed by the function of the divergent integral e.g. \( \int_0^\infty g(\alpha) d\alpha \), the domain of the integral is cut off as the following: \( \int_0^M g(\alpha) d\alpha \) or \( \int_M^\infty g(\alpha) d\alpha \) or \( \int_0^M g(\alpha) d\alpha \) etc. After these cut-off procedures the physical quantity \( \mathcal{M} \left( \int_0^\infty g(\alpha) d\alpha \right) \) is frequently calculated as the limit \( \mathcal{M} \left( \lim_{M \to \infty} \int_0^M g(\alpha) d\alpha \right) \) etc.

To express these procedures using the corresponding states of field, one defines cut-off operator \( P_N \) which maps a vector \( \Pi_{t=1}^\infty f_t \) to the vector \( (\Pi_{t=1}^{m-1} f_t) \otimes (\Pi_{t=m+1}^\infty f_t) \) where \( e_{e_0} \) is a vector which corresponds to the occupation number e.g. 0 for the quantum number \( \alpha \).

In some cases we may be able to consider that the cut-off proceeds as follows e.g.

\[ \lim_{M \to \infty} \int_0^M g(\alpha) d\alpha = \lim_{N \to \infty} \langle \Phi | G | P_N \Phi \rangle = \langle \Phi | G | \Psi \rangle = \int_0^\infty g(\alpha) d\alpha. \]

4. Now we consider the ambiguity of these cut-off procedures by the following four examples.

Example 1. For any linear aggregate of \( c \)-sequences \( \Phi \) there exists a set of vectors \( \Phi^a \) of \( H \) such that \( P_n \Phi^a \) converges to 0.

Proof. For \( \Phi = \Sigma_{\nu=1}^n \Pi_{\alpha=1}^\infty f_{\alpha \nu} = \Sigma_{\nu=1}^n (\Pi_{\alpha \in N} \otimes \varepsilon_{\alpha \nu} \otimes \Pi_{\beta > N} \delta_{\beta \nu} f_{\beta \nu}) \), we can take \( \varepsilon_{\alpha \nu} \delta_{\beta \nu} \) such that the equalities \( (\Pi \otimes \varepsilon_{\alpha \nu} ) \otimes (\Pi \otimes \delta_{\beta \nu} ) = 1 \) and \( (\Pi_{\alpha \leq N} \otimes \varepsilon_{\alpha \nu} N f_{\alpha \nu} ) \downarrow 0 \) hold. Let \( \Phi^N = \Sigma_{\nu=1}^n (\Pi_{\alpha \leq N} \otimes \varepsilon_{\alpha \nu} \otimes \Pi_{\beta > N} \delta_{\beta \nu} f_{\beta \nu}) \), then \( P_N \Phi^N = \Sigma_{\nu=1}^n (\Pi_{\alpha \leq N} \otimes \varepsilon_{\alpha \nu} f_{\alpha \nu} \otimes \Pi_{\beta > N} \delta_{\beta \nu} f_{\beta \nu} ) \) converges to 0, since \( \|P_N \Phi^N\| \leq \Sigma_{\nu=1}^n \|\Pi_{\alpha \leq N} \otimes \varepsilon_{\alpha \nu} f_{\alpha \nu}\| < \varepsilon \). This sort of ambiguity causes form the property (1).

To avoid this sort of ambiguity one may immediately think about the standard forms: \( (\Pi_{\alpha \leq N} f_{\alpha \nu}) \cdot \Pi_{\alpha \nu} \otimes (f_{\alpha \nu} || f_{\alpha \nu} ) \) for \( \Phi = \Pi \otimes f_{\alpha \nu} \) and \( \Sigma_{\nu=1}^n (\Pi_{\alpha \leq N} f_{\alpha \nu} || f_{\alpha \nu} || f_{\alpha \nu} ) \) for \( \Phi = \Sigma_{\nu=1}^n \Pi_{\alpha \nu} f_{\alpha \nu} \), where \( \alpha = 1, 2, 3, \cdots \) and \( \nu = 1, 2, \cdots, n \).

One can see however even these standard forms do not always give unique cut-off vector because of the property (2) as the follow-
Example 2. Assume that $\varphi = \Pi \otimes f + \Pi \otimes g$ satisfies the following two conditions: (1) $||f|| = ||g|| = 1$ (2) $\langle f, g \rangle = 0$ for any $\alpha$. Then we see that $\Pi \otimes f = \psi_N \otimes \psi_{\infty}$, $\Pi \otimes g = \Phi_N \otimes \Phi_{\infty}$ and $\psi_N \perp \Phi_N$, $\psi_{\infty} \perp \Phi_{\infty}$, $||\Phi_N|| = ||\Phi_{\infty}|| = ||\psi_N|| = ||\psi_{\infty}|| = 1$.

Now

$\varphi = \psi_N \otimes \psi_{\infty} + \Phi_N \otimes \Phi_{\infty} = \psi_N \otimes (\psi_{\infty} - \lambda \Phi_{\infty}) + (\lambda \psi_N + \Phi_N) \otimes \Phi_{\infty}$.

Hence using the first expression,

$P_N \varphi = \psi_N \otimes e_{\infty} + \Phi_N \otimes e_{\infty} = (\psi_N + \Phi_N) \otimes e_{\infty}$.

So $||P_N \varphi|| = \sqrt{2}$. Using the 2nd expression, however,

$\varphi = ||\psi_{\infty} - \lambda \Phi_{\infty}|| \cdot \psi_N \otimes \frac{(\psi_{\infty} - \lambda \Phi_{\infty})}{||\psi_{\infty} - \lambda \Phi_{\infty}||} + ||\lambda \psi_N + \Phi_N|| \cdot \frac{(\lambda \psi_N + \Phi_N)}{||\lambda \psi_N + \Phi_N||} \otimes \Phi_{\infty}$.

Hence

$P_N \varphi = \{||\psi_{\infty} - \lambda \Phi_{\infty}|| + \lambda) \psi_N + \Phi_N \} \otimes e_{\infty}$.

Hence

$||P_N \varphi|| = \sqrt{(1 + \lambda^2 + \lambda^2)} = \sqrt{2(1 + 2\lambda(1 + \lambda^2 + \lambda^2)}$. So $||P_N \varphi||$ runs through the value from $\sqrt{2}$ to $\infty$. We can see also here

$\lim_{\lambda \to \infty} P_N \varphi = \psi_N \otimes e_{\infty}$ and $\lim_{\lambda \to \infty} P_N \varphi = \psi_N \otimes e_{\infty}$.

The following two examples show more pathological ambiguity of cut-off operators.

Example 3. For any state $\Phi$ whose corresponding vector is a $c$-sequence, there exist standard expressions $\Phi^N$ such that $P_N \Phi^N$ converges to 0.

Example 4. For any $\Phi, \psi$ which are expressed by finite linear aggregate of $c$-sequence, there exist sets of expressions $\{\Phi^N\}$ and $\{\psi^N\}$ such that $P_N \Phi^N = P_N \psi^N$.

Proof. Let $c$-sequence $\Phi = \Pi \otimes f = (\Pi^N \otimes f) \otimes (\Pi^\infty \otimes f) = \Phi_N \otimes \Phi_{\infty}$ be a standard form. Let $e_{o\omega}, e_{o\omega}$ be a unit vector such that $e_{o\omega} \perp e_{o\omega}$ and $||e_{o\omega}|| = ||e_{o\omega}|| = 1$ and $\Phi_{\infty} = \frac{1}{\sqrt{2}} (e_{o\omega} - e_{o\omega})$. Then we see that $\Phi = \Phi^N$

$= \Phi_N \otimes \left(\frac{1}{\sqrt{2}} e_{o\omega} - \frac{1}{\sqrt{2}} e_{o\omega}\right) = \frac{1}{\sqrt{2}} (\Phi_N \otimes e_{o\omega}) - \frac{1}{\sqrt{2}} (\Phi_N \otimes e_{o\omega})$.

Hence

$P_N \Phi = \frac{1}{\sqrt{2}} (\Phi_N \otimes e_{o\omega}) - \frac{1}{\sqrt{2}} (\Phi_N \otimes e_{o\omega}) = 0$.

In case $\Phi = \Sigma_i c_i \Pi^N \otimes f_{o\omega}$, we can prove similarly.

Example 4 is also easily constructed utilizing Example 3.

Lastly it is remarkable that the cut-off process can not always be done in the space $H$ as the following example shows:

Example 5. Let $\Phi = \Sigma_i c_i \Pi^N \otimes \psi_{o\omega} \otimes \psi_{o\omega} \otimes \Pi^\infty \otimes f_{o\omega}$ where $c_i > 0$, $\sum c^2_i < \infty$ and $\sum c_i = \infty$ are satisfied. Then we can see easily that $||\Phi|| < \infty$ and $||P_N \Phi|| = \infty$. 

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5. In view of these ambiguities one may ask the following questions: Under what expressions the customary cut-off procedure is done?

Some answers for these questions will be given in the following articles [1] [3] which also contain some sort of remedies against these ambiguities.

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References