3. Certain Embedding Problems of Semigroups

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1. By a *left translation* of a semigroup S we mean a transformation λ of $S, x\to x\lambda$, satisfying $(xy)\lambda=(x\lambda)y$, for all x, y in S. A *right translation* of S is a transformation ρ satisfying $(xy)\rho=x(y\rho)$, for all x, y in S. A left translation λ and a right translation ρ are said to be *linked* if $x(y\lambda)=(x\rho)y$, for all x, y in S. We note that for each a in S, the transformation λ_a defined by $x\lambda_a=ax$, for all x in S, is a left translation of S, the transformation ρ_a defined by $x\rho_a=xa$, for all x in S, is a right translation of S, and λ_a and ρ_a are linked. We call λ_a an *inner left translation* of S, ρ_a an *inner right translation* of S. A semigroup S is said to be *weakly reductive* if, for any a, b in S, $ax=bx$ and $xa=xb$, for all x in S, imply $a=b$.

It was proved in [1] that a weakly reductive semigroup S can be embedded into a semigroup T so that

(1) S is an ideal of T,

(2) every left translation of S is induced by some inner left translation of T, and every right translation of S is induced by some inner right translation of T,

if and only if

(3) every left translation of S is linked with some right translation of S, and *vice versa.*

However, the general case in which weak reductivity is not assumed was open. In this paper we shall give necessary and sufficient conditions for an arbitrary semigroup S so that it can be embedded into a semigroup T with the properties (1) and (2). The special case for weakly reductive semigroups will follow as a corollary. We shall also discuss the embedding of a semigroup S into a semigroup T under conditions somewhat weaker than (1) and (2).

2. The open problem in [1] can be solved as follows:

Theorem 1. A semigroup S can be embedded into a semigroup T so that

(1) $ST\subseteq S, TS\subseteq S$,

(2) for every left translation λ of S there exists a in T such that $x\lambda=ax$, for all x in S, and for every right translation ρ of S there exists b in T such that $x\rho=xb$, for all x in S,

if and only if

(3) every left translation of S is linked with some right translation of S, and *vice versa,*

(4) every left translation of S commutes with every right translation of S.

Proof. Assume S embedded into T under conditions (1) and (2). Let λ be a left translation of S. Then, by (2), there exists a in T such that $\lambda=\lambda_a|S$ (the restriction of λ_a to S). Since $ST\subseteq S$, $\rho_a|S$ is a right translation of S, and hence $\rho_a|S$ is the right translation of S linked with $\lambda=\lambda_a|S$. Similarly, every right translation of S is linked with some left translation of S. Therefore (3) holds. Now let λ be a left translation of S, ρ a right translation of S. By (2) there exist a, b in T such that for all x in S, $x\lambda=ax$, $x\rho=xb$. Thus, $x\rho\lambda=a(xb)=(ax)b=x\lambda\rho$. Conversely, suppose S satisfies (3) and (4). Let T_0 be the translational hull of S in [1], that is, the semigroup of all pairs (λ, ρ) of linked left and right translations λ and ρ of S under the operation $\circ$ defined by $(\lambda, \rho)\circ(\lambda', \rho')=(\lambda'\lambda, \rho\rho')$. Now let $T=S\cup T_0$ and define an operation $*$ on T by

(5) $x*y=xy$

(6) $x*(\lambda, \rho)=x\rho$

(7) $(\lambda, \rho)*x=x\lambda$

(8) $(\lambda, \rho)*(\lambda', \rho')=(\lambda, \rho)\circ(\lambda', \rho')$

where x, y in S, (λ, ρ), (λ', ρ') in T_0. Clearly, S is embedded into T so that $S*T\subseteq S$, $T*S\subseteq S$, by (5), (6), and (7). Also, by condition (3), for every left translation λ of S there exists at least one element (λ, ρ) of T such that $x\lambda=(\lambda, \rho)*x$, for all x in S, and for every right translation of S, the analogous statement holds. Thus S would be embedded into a semigroup T under conditions (1) and (2), provided $*$ is associative on T. We shall prove $\alpha*(\beta*\gamma)=(\alpha*\beta)*\gamma$ where α, β, γ are arbitrary elements of $T=S\cap T_0$. Since $*$ coincides with the operations on S, T_0, the associativity holds, by (5) and (8), when α, β, γ in S or when in T_0. The verification of the remaining cases is straightforward from the following facts: λ is a left translation of S, ρ is a right translation of S, the mapping $(\lambda, \rho)\rightarrow\rho$ is a homomorphism, $(\lambda, \rho)\rightarrow\lambda$ is a dual homomorphism, λ is linked with ρ, and (4) holds (cf. p. 3 of [3]).

Corollary 1. Let S be a weakly reductive semigroup. Then S can be embedded into a semigroup T under conditions (1) and (2) if and only if (3) holds.

Proof. It suffices to show that for a weakly reductive semigroup S, (3) implies (4). Assume (3). Let λ be a left translation of S, ρ a right translation of S, $\bar{\lambda}$ a right translation of S linked with λ, $\bar{\rho}$ a left translation of S linked with ρ. Let x, y be arbitrary elements of S. Then

$$(y\rho\lambda)x=((y\rho)x)\lambda=(y(x\bar{\rho}))\lambda=(y\lambda)(x\bar{\rho})=(y\lambda\rho)x,$$

$$x(y\rho\lambda)=(x\bar{\lambda})(y\rho)=((x\bar{\lambda})y)\rho=(x(y\lambda))\rho=x(y\lambda\rho).$$

Since S is weakly reductive, $y\rho\lambda=y\lambda\rho$ for arbitrary y. Therefore, (4) holds.

Corollary 2. If S is a commutative semigroup, (1) and (2) are equivalent to

(4′) every translation of S commutes with any other translation of S.

If S satisfies $S^2=S$, then (1) and (2) are equivalent to (3), (cf. Exercise, p. 13 in [1]). Accordingly, if S is commutative and $S^2=S$, S is always embeddable in T.

3. We can weaken the condition (2) in the following two ways:

(9) Every translation of S, whether left or right, is induced by an inner left or right translation of T (possibly by both inner left and right translations of T).

(10) Every right translation of S is induced by inner right translation of T.

Theorem 2. A semigroup S can be embedded into a semigroup T so that

(1) $ST\subseteq S$, $TS\subseteq S$,

(9) for every left or right translation τ of S there exists t in T such that $x\tau=tx$ for all x in S, or $x\tau=xt$ for all x in S,

if and only if there exist a subsemigroup A of the semigroup of all left translations on S and a subsemigroup B of the semigroup of all right translations on S such that

(11) each element of A is linked with some element of B and *vice versa*,

(12) every element of A commutes with every element of B,

(13) every translation of S belongs to A or to B (possibly to both A and B).

Proof. Assume (1) and (9). Let A be the subset of all left translations of S which are induced by inner left translations. Then A is a subsemigroup of left translations of S. Similarly, the set B of all right translations of S which are induced by inner right translations is a subsemigroup of right translations of S. By the definition of A and of B, (9) implies (13) immediately. (11) and (12) are obtained in the same way as the proof of (3) and (4) in Theorem 1. The proof of the converse parallels closely the proof of the converse of Theorem 1. Assume there exist sets A and B satisfying (11), (12), and (13), and let $T_0=\{(\alpha, \beta);\ \alpha\in A, \beta\in A$, and α are linked with $\beta\}$. We may define an operation $*$ on T in the same way as (5) through (8).

Theorem 3. A semigroup S can be embedded into a semigroup T so that

(1) $ST\subseteq S$, $TS\subseteq S$,

(14) for every right translation ρ of S there exists t in T such that $x\rho=xt$ for all x in S,

if and only if there exists a subsemigroup A of the semigroup of all left translations on S such that

(15) every right translation of S is linked with some element of A and *vice versa*;

(16) every element of A commutes with every right translation of S.

The proof is similar to those of Theorems 1 and 2.

4. For simplicity, the embeddings in the cases of Theorems 1, 2, and 3 are called those in "two-sided way", "mixed way" and "right-sided way", respectively. The dual case of Theorem 3 is called "left-sided way"; and "one-sided way" means either "left-sided way" or "right-sided way". Clearly, the embeddability in two-sided way implies that in mixed way and in one-sided way.

Theorem 4. A semigroup S can be embedded in the two-sided way if and only if S can be embedded in the mixed way and S satisfies (4).

Proof. We may show that (3) is fulfilled if S is embeddable in the mixed way. If ρ is a right translation and at the same time a left translation of S, ρ is linked with itself. Even if ρ is a right translation but not a left translation, there is a left translation λ which is linked with ρ because of the assumption of the mixed way. The same thing holds with respect to any left translation which is not a right translation. Thus (3) has been proved and hence this theorem has been proved by Theorem 1.

Corollary 3. Let S be a weakly reductive semigroup. S can be embedded in the two-sided way if and only if S can be embedded in the mixed way.

Proof. If S is embeddable in the mixed way, (3) is satisfied, as seen in the proof of Theorem 4. By Corollary 1, S is embeddable in the two-sided way.

Example 1. A null-semigroup of order >2 (cf. [1]) is an example of S which is not embeddable in the two-sided way but embeddable in the mixed way or in the one-sided way.

Example 2. Consider a semigroup S of order 4 defined as follows:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	a	b

There is no distinction between right and left translations. All those are given as follows (cf. [2]):

$$\begin{pmatrix} a & b & c & d \\ a & a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & a & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}.$$

Since they satisfy (3) and (4), S can be embedded in the two-sided way, for example:

	a	*b*	*c*	*d*	*e*	*f*	*g*
a	*a*	*a*	*a*	*a*	*a*	*a*	*a*
b	*a*	*a*	*a*	*a*	*a*	*a*	*b*
c	*a*	*a*	*b*	*b*	*a*	*a*	*c*
d	*a*	*a*	*a*	*b*	*a*	*b*	*d*
e	*a*	*a*	*a*	*a*	*e*	*e*	*e*
f	*a*	*a*	*b*	*a*	*e*	*e*	*f*
g	*a*	*b*	*c*	*d*	*e*	*f*	*g*

The following embedding of S is in the mixed way, but this embedding is neither in the two-sided way nor in the one-sided way.

	a	*b*	*c*	*d*	*e*
a	*a*	*a*	*a*	*a*	*a*
b	*a*	*a*	*a*	*a*	*b*
c	*a*	*a*	*b*	*b*	*c*
d	*a*	*a*	*a*	*b*	*d*
e	*a*	*b*	*c*	*d*	*e*

The following is in the right-sided way, and not in the left-sided way, but in the mixed way; and hence not in the two-sided way.

	a	*b*	*c*	*d*	*e*	*f*	*g*
a	*a*	*a*	*a*	*a*	*a*	*a*	*a*
b	*a*	*a*	*a*	*a*	*a*	*a*	*b*
c	*a*	*a*	*b*	*b*	*a*	*a*	*c*
d	*a*	*a*	*a*	*b*	*a*	*b*	*d*
e	*a*	*a*	*a*	*a*	*e*	*e*	*e*
f	*a*	*a*	*a*	*a*	*e*	*e*	*f*
g	*a*	*b*	*c*	*d*	*e*	*f*	*g*

Example 3. Let S be a semigroup of order 3 defined by

	a	*b*	*c*
a	*a*	*b*	*a*
b	*a*	*b*	*a*
c	*a*	*b*	*a*

S can be embedded in the right-sided way:

	a	b	c	d	e	f	g	h	i
a	a	b	a	a	a	a	b	b	a
b	a	b	a	a	a	b	b	a	b
c	a	b	a	a	c	a	b	b	c
d	a	b	a	d	d	d	g	g	d
e	a	b	a	d	e	d	g	g	e
f	a	b	a	d	d	f	g	h	f
g	a	b	a	d	d	g	g	d	g
h	a	b	a	d	d	h	g	f	h
i	a	b	a	d	e	f	g	h	i

The left translation $\begin{pmatrix} a & b & c \\ a & b & d \end{pmatrix}$ of S is not linked with any right translation of S, and hence S can not be embebbed in the mixed way, nor in the left-sided way.

There still remain a few problems:

Problem 1. If S can be embedded in the mixed way, can S be embedded in the right-sided or left-sided way?

Problem 2. For a given S, is there the smallest or a minimal extension in which S can be embedded in each of the three ways, if it exists? What relationship is there between them?

Problem 3. Can any semigroup be embeddable either in the right-sided way or in the left-sided way?

References

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[3] ——: The theory of construction of finite semigroups. II. Osaka Math. J., **9**, 1-42 (1957).