22. Construction of Finite Commutative z-Semigroups

By Miyuki YAMADA
Department of Mathematics, Shimane University
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§ 1. **Introduction.** As defined by Tamura [4], a semigroup is called a *z-semigroup* if it has a zero element, 0, but has no idempotent except 0. In particular, for a finite commutative semigroup S it is easily seen that S is a z-semigroup if and only if it satisfies the following two conditions:

(1) S has a zero element 0

and (2) $S \supset S^2 \supset \cdots \supset S^p = \{0\}$ for some positive integer p.[1]

If $S \backslash S^2$ is non-empty, every element of $S \backslash S^2$ is called a *prime element* of S.

In the case of $p=1$ or $p=2$, S satisfies the following

(3) $S=\{0\}$

or (4) $xy=0$ for any $x, y \in S$,

respectively.

Such a semigroup S is called a *trivial z-semigroup* or a *null semigroup*, corresponding to $p=1$ or $p=2$.

Now, the problem of construction of finite commutative z-semigroups occupies an important part in the problem of construction of finite commutative semigroups. In this paper, we shall deal with this problem and present a method of constructing all possible commutative z-semigroups of a given order. The proofs are omitted and will be given in detail elsewhere.[2]

§ 2. **Commutative z-semigroups of order n.** At first, we have

Theorem 1. *For any positive integer n, there exists a commutative z-semigroup of order n.*

Let G be a semigroup with a zero element 0. The subset A of G, where $A=\{x : x \in G, xy=yx=0 \text{ for all } y \in G\}$, is a subsemigroup of G. We shall call A the *annihilator* of G.

Lemma 1. *The annihilator of a non-trivial, finite commutative z-semigroup has a non-zero element (see also Tamura [3]).*

Lemma 2. *Let S be a commutative z-semigroup of order $n+1$ ($n \geqq 1$). Let 0 be the zero element of S and let u be a non-zero element contained in the annihilator of S. Then the set $\{0, u\}$ is both a null subsemigroup and an ideal of S, and the factor semigroup $D=S/\{0, u\}$ of S mod $\{0, u\}$ in the sense of Rees [2] is a commutative z-semi-*

1) $A \supset B$ means 'B is a proper subset of A'.

2) This is an abstract of a paper which will appear elsewhere.

group of order n. Further, in this case S is a commutative extension of $\{0, u\}$ *by D in the sense of Clifford* [1].[3] *Accordingly, S is a commutative extension of a null semigroup of order 2 by a commutative z-semigroup of order n.*

Conversely, we have

Lemma 3. *A commutative extension of a null semigroup of order 2 by a commutative z-semigroup of order n is a commutative z-semigroup of order* $n+1$.

Remark. For any given null semigroup N of order 2 and for any commutative z-semigroup Z of order n, existence of a commutative extension of N by Z is proved by the following example: Let $N=\{0, u\}$, where 0 is the zero element of N. Let $\mathbf{0}$ be the zero element of Z, and put $S=Z\backslash\mathbf{0}+\{0, u\}$.

Then S becomes a commutative extension of N by Z by the multiplication $\circ$ defined as follows:

$$x \circ y=\begin{cases} xy & \text{if } x, y\in Z \text{ and } xy\neq\mathbf{0}, \\ 0 & \text{otherwise.}\end{cases}$$

Combining Lemmas 2 and 3, we obtain the following

Theorem 2. *A commutative z-semigroup of order* $n+1$ $(n\geqq 1)$ *is a commutative extension of a null semigroup of order 2 by a commutative z-semigroup of order n, and vice-versa.*

Now, we consider the problem:

(A) Construct all possible commutative z-semigroups of order n for a given positive integer n.

For $n=1$ or 2 this problem is easily solved, since a commutative z-semigroup of order 1 or 2 is a trivial z-semigroup or a null semigroup respectively.

Hence, the problem (A) is reduced to the following problem:

(B) We assume that we can construct all possible commutative z-semigroups of order m $(m\geqq 2)$. Construct all possible commutative z-semigroups of order $m+1$.

Further, by Theorem 2 the problem (B) is reduced to the following problem:

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3) Let K be a semigroup. Let L be a semigroup with a zero element 0, having no element in common with K. Let $M=K+L\backslash\{0\}$.

If a binary operation $\circ$ in M satisfies the following

$$(\mathrm{M}) \quad \begin{cases} (1) \quad x\circ y \begin{cases} =xy & \text{if } x, y\in K \text{ or if } x, y\in L \text{ and } xy\neq 0, \\ \in K & \text{otherwise,}\end{cases} \\ (2) \quad (x\circ y)\circ z=x\circ(y\circ z),\end{cases}$$

then the resulting system $M(\circ)$ becomes a semigroup, which is called an *extension* of K by L. If K and L are commutative, we can consider the case in which $M(\circ)$ becomes a commutative semigroup. In this case, we shall call $M(\circ)$ a *commutative extension* of K by L.

(C) Construct all possible commutative extensions of a given null semigroup of order 2 by a given commutative z-semigroup of order m ($m \geqq 2$).

We shall deal with this problem (C) in the next paragraph.

Remark. The whole discussion and result of this paragraph still hold, even if we substitute the terms 'z-semigroup' and 'extension' for the terms 'commutative z-semigroup' and 'commutative extension' respectively. Accordingly, the problem of constructing all finite z-semigroups is reduced to the following problem:

(C*) Construct all possible extensions of a given null semigroup of order 2 by a given z-semigroup of order m ($m \geqq 2$).

§ 3. **C-factors of a finite commutative z-semigroup.** Let N be a null semigroup of order 2, and put $N=\{0, z\}$, where 0 is the zero element of N. Let T be a finite commutative z-semigroup having $\mathbf{0}$ as its zero element. Let $T^*=T\setminus\{\mathbf{0}\}$, and let $S=N+T^*$. Let $\Omega=\{(x, y): xy=\mathbf{0},\ x, y \in T\}$.

Then, any subset Λ of Ω satisfying the following conditions (1)–(3) is an ideal of the direct product $T \times T$:

$$\text{(E)} \quad \begin{cases} (1) & (t, \mathbf{0}) \in \Lambda \text{ for any } t \in T, \\ (2) & (tv, w) \in \Lambda \text{ implies } (t, vw) \in \Lambda, \\ \text{and } (3) & (v, w) \in \Lambda \text{ implies } (w, v) \in \Lambda. \end{cases}$$

Such a Λ is called a *commutative extension factor* (abbrev., *C-factor*) of T. It is easy to see that Ω itself is the greatest C-facter of T.

Under this definition, we have

Theorem 3. *Let Λ be a C-factor of T, and define multiplication $\circ$ in S by the following*

$$\text{(P)} \quad x \circ y = \begin{cases} xy & \text{if } x, y \in T^*,\ xy \neq \mathbf{0} \text{ or if } x, y \in N \\ 0 & \text{if } x \in N \text{ or } y \in N \\ 0 & \text{if } (x, y) \in \Lambda,\ x, y \in T^* \text{ and } xy=\mathbf{0} \\ z & \text{if } (x, y) \notin \Lambda,\ x, y \in T^* \text{ and } xy=\mathbf{0}. \end{cases}$$

Then, $S(\circ)$ becomes a commutative extension of N by T. Further, every commutative extension of N by T is found in this fashion.

By Theorem 3, the problem of determining all commutative extensions of N by T is reduced to the problem of finding all C-factors of T. Next, we shall consider this problem.

Theorem 4. *Let*

$$\Gamma=\{(t_1t_2, t_3): t_1, t_2, t_3 \in T,\ t_1t_2t_3=\mathbf{0}\} \cup \{(t_1, t_2t_3): t_1, t_2, t_3 \in T,\ t_1t_2t_3=\mathbf{0}\}.$$

Then,

(1) *Γ is a C-factor of T,*

(2) *$\Gamma=\Omega\setminus\{(x, y): x, y$ are prime elements of $T\}$*

and (3) *if $\Omega \supseteqq \Lambda \supseteqq \Gamma$ and if Λ satisfies the condition* (3) *of* (C), *then Λ is a C-factor of T.*

If a sequence $\mathfrak{S}=\{t, t_0, t_1, t_2, t_3, \cdots, t_r\}$ of elements of T, where r

is an even integer $\geqq 2$, satisfies

(1) $(t, t_0)\in\Omega$

and (2) $t=t_1t_2, t_0t_1=t_3t_4, t_2t_3=t_5t_6, \cdots, t_{r-4}t_{r-3}=t_{r-1}t_r$

($t=t_1t_2$ in the case of $r=2$),

then $\mathfrak{S}$ is called a (t, t_0)-*chain* (in T). Further, in this case the ordered set (t_{r-2}, t_{r-1}, t_r) is called the *final part* of $\mathfrak{S}$. It should be noted that for a given $(t, t_0)\in\Omega$ such a (t, t_0)-chain is not necessarily unique even if it exists.

Lemma 4.

(1) *If* (t_0, t_1, t_2) *is the final part of a* $(t, \mathbf{0})$*-chain*, $\{t, t_0, t_1, t_2\}$, *then* $(t_0t_1, t_2)=(\mathbf{0}, t_2)$ *and* $(t_2, t_0t_1)=(t_2, \mathbf{0})$.

(2) *If* (t_{r-2}, t_{r-1}, t_r) *is the final part of a* $(t, \mathbf{0})$*-chain and if* $r\geqq 4$, *then* (t_{r-2}, t_{r-1}, t_r) *is also the final part of some* $(\mathbf{0}, t')$*-chain.*

Using this lemma, we have

Theorem 5. *The least C-factor* Λ_0 *of* T *is as follows:*

$\Lambda_0=\{(v, \mathbf{0}) : v\in T\}\cup\{(\mathbf{0}, w) : w\in T\}\cup\{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ *is the final part of a* $(\mathbf{0}, t)$*- or* $(t, \mathbf{0})$*-chain for some* $t\in T\}\cup\{(t_r, t_{r-2}t_{r-1}) :$ (t_{r-2}, t_{r-1}, t_r) *is the final part of a* $(\mathbf{0}, t)$*- or* $(t, \mathbf{0})$*-chain for some* $t\in T\}$ $=\{(v, \mathbf{0}) : v\in T\}\cup\{(\mathbf{0}, w) : w\in T\}\cup\{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ *is the final part of a* $(\mathbf{0}, t)$*-chain for some* $t\in T\}\cup\{(t_r, t_{r-2}, t_{r-1}) : (t_{r-2}, t_{r-1}, t_r)$ *is the final part of a* $(\mathbf{0}, t)$*-chain for some* $t\in T\}$.

Further, we have the following

Theorem 6. *Let* Δ_0 *be a C-factor of* T *and let* (u, v) *be an element of* Ω. *Then, the C-factor* Δ *of* T *generated by* $\{\Delta_0, (u, v)\}$, *that is, the least C-factor containing* Δ_0 *and* (u, v) *is as follows:*

$\Delta=\Delta_0\cup\{(u, v)\}\cup\{(v, u)\}\cup\{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ *is the final part of a* (u, v)*- or* (v, u)*-chain*$\}\cup\{(t_r, t_{r-2}t_{r-1}) : (t_{r-2}, t_{r-1}, t_r)$ *is the final part of a* (u, v)*- or* (v, u)*-chain*$\}$.

For any C-factor Λ of T and for any subset Ξ of Ω, let $\Gamma(\Lambda, \Xi)$ be the least C-factor of T containing Λ and Ξ. Put $\{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_r\}$ $=\Xi$, where $\alpha_j\in\Omega$. Then, we can easily prove the following relation:

$$\Gamma(\Gamma(\cdots\Gamma(\Gamma(\Gamma(\Lambda, \{\alpha_1\}), \{\alpha_2\}), \{\alpha_3\}), \cdots, \{\alpha_{r-1}\}), \{\alpha_r\})=\Gamma(\Lambda, \Xi).$$

Now, by Theorems 5 and 6 we can obtain all C-factors of T. In fact: $\{\Gamma(\Lambda_0, \Sigma) : \Sigma\subset\Omega\backslash\Lambda_0\}$ is the totality of all C-factors of T.

Remark. In the case in which T is not necessarily commutative, we can also introduce the concept of E-factors of T as follows: A subset Π of Ω satisfying the condition

(E*) { (1) $(t, \mathbf{0})\in\Pi$ and $(\mathbf{0}, t)\in\Pi$ for any $t\in T$
and (2) $(tv, w)\in\Pi$ implies $(t, vw)\in\Pi$, and $(t, vw)\in\Pi$ implies $(tv, w)\in\Pi$

is called an *extension factor* (abbrev., *E-factor*) of T. It is clear that Ω itself is the greatest E-factor of T. Let $\Theta=\{(x, y) : x, y\in T\}$. Define multiplication $\circ$ (◎) in Θ as follows:

$$(x, y) \circ (v, w) = (xv, wy) \quad ((x, y) \circledcirc (v, w) = (vx, yw)).$$

Then, the resulting system $\Theta(\circ)$ ($\Theta(\circledcirc)$) becomes a semigroup. It is clear that any E-factor of T is a left ideal of $\Theta(\circ)$ and a right ideal of $\Theta(\circledcirc)$. Also, it is easily seen that both $\Theta(\circ)$ and $\Theta(\circledcirc)$ coincide with $T \times T$ if T is commutative. Hence, if T is commutative any E-factor of T is an ideal of $T \times T$. Every C-factor of a finite commutative z-semigroup is an E-factor, but the converse is not true.

We have

Theorem. *An E-factor Λ of a finite commutative z-semigroup is a C-factor if and only if it satisfies the condition* (3) *of* (E).

Finally, we obtain the following extension theorem for the case in which T is not necessarily commutative:

Theorem. *Let Λ be an E-factor of T, and define multiplication $\circ$ in S by* (P) *of Theorem* 3. *Then, the resulting system $S(\circ)$ becomes an extension of N by T. Further, every extension of N by T is found in this fashion.*

References

[1] A. H. Clifford: Extensions of semigroups. Trans. Amer. Math. Soc., **68**, 165–173 (1950).

[2] D. Rees: On semigroups. Proc. Cambridge Phil. Soc., **36**, 387–400 (1940).

[3] T. Tamura: On finite one-idempotent semigroups (1). Jour. Gakugei Tokushima Univ., **4**, 11–20 (1954).

[4] ——: Note on finite semigroups and determination of semigroups of order 4. Jour. Gakugei Tokushima Univ., **5**, 17–27 (1954).