

20. On Postulate-Sets for Newman Algebra and Boolean Algebra. II

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The present paper treats of the latter part of the second paragraph, on the construction of some independence systems, and also, of the third paragraph, the independence proofs for the four new sets for Boolean algebras. For the notations and the general assumptions see the preceding article.

We go further to construct those systems which can be used for the independence proofs of postulates B_1 , B'_1 in Set II_1 and Set II_2^* respectively. We shall begin with the independence proof of postulate B_1 (or B'_1) in Set II_2^* .

Definition 4. By a "Generalized quasi-Boolean ring of type R-D (or L-D)" we shall mean an algebraic system M with the operations for addition and multiplication such that M is an abelian group in which $a+a=0$ for all a , and $a(ab)=(aa)b$, $aa=a$, $a0=0$ (or $0a=0$), and $(a+b)c=ac+bc$ (or $a(b+c)=ab+ac$) for all a, b, c . And by "quasi-Boolean ring of type R-D (or L-D)" we shall mean a generalized quasi-Boolean ring of type R-D (or L-D) with "unit", 1, for which $a1=1a=a$ holds for all a .

Theorem 11. Let M be a generalized quasi-Boolean ring of type R-D (or L-D) in which left (or right) distributive law does not hold. If M is finite it has a cardinal number 2^n , $n \geq 2$.

Proof. By the definition M is an addition abelian group in which $a+a=0$ holds. And by the hypotheses, that M is finite and that the left (or right) distributive law does not hold in M , it is clear that M has a cardinal number 2^n , $n \geq 2$.

Theorem 12. Every generalized quasi-Boolean ring M of type R-D (or L-D), can be imbedded in a quasi-Boolean ring P of type R-D (or L-D), in such a manner that P is unique in the following sense: if Q is a quasi-Boolean ring of type R-D (or L-D) containing M , then Q contains also a quasi-Boolean ring P^* of type R-D (or L-D) isomorphic to P and containing M .

Proof. We shall give a brief proof; of course we may disregard the trivial case of this theorem where M has a unit and P coincides with M . Providing an abstract element ε , distinct from those of M , we shall define

$$\varepsilon\varepsilon=\varepsilon, \quad \varepsilon m=m\varepsilon=m, \quad \varepsilon+0=0+\varepsilon=\varepsilon, \quad \varepsilon+\varepsilon=0,$$

where 0 being the zero element and m any element of M . Now we let P the direct union of $M \times I$ of pairs (m, α) where $m \in M$ and α is an element 0 or ε of two-element Boolean ring I . We can define the equality, addition and multiplication of (m, α) and (n, β) in P , in the same manner as we have defined in S for Theorem 10. And under these operations, it is easy to see that P is a quasi-Boolean ring of type R-D (or L-D) with unit $(0, \varepsilon)$. To prove the "uniqueness" of P we can follow the proof of Stone for his Theorem 1 [2: pp. 40-42], as the left (or right) distributive law in M is used only to show that the same law holds in P .

Now let us try to construct a non left (or right) distributive generalized quasi-Boolean ring of type R-D (or L-D). If such a ring exists, it has by Theorem 11 a cardinal number 2^n , $n \geq 2$, and the additive group of type $(2, 2, \dots, 2)$. So if we construct such a ring with four elements, it is a "smallest" one in the sense that it can not have any proper subring with the same property. Now we have the following example of a generalized quasi-Boolean ring of type R-D with four elements without unit and non left-distributive.

| | |
|---|---------|
| + | 0 a b c |
| 0 | 0 a b c |
| a | a 0 c b |
| b | b c 0 a |
| c | c b a 0 |

| | |
|---|---------|
| × | 0 a b c |
| 0 | 0 0 0 0 |
| a | 0 a 0 0 |
| b | 0 0 b c |
| c | 0 a b c |

Thus we have an example of "smallest" non left-distributive generalized quasi-Boolean ring of type R-D. We shall denote this ring with M . Applying Theorem 12 to this M , we obtain a system P of eight elements as follows:

| | | | | | |
|----------|-------------------------------------|----------|---|----------|----------|
| + | 0 1 a b c α β γ | × | 0 1 a b c α β γ | a | a' |
| 0 | 0 1 a b c α β γ | 0 | 0 0 0 0 0 0 0 0 | 0 | 1 |
| 1 | 1 0 α β γ a b c | 1 | 0 1 a b c α β γ | 1 | 0 |
| a | a α 0 c b 1 γ β | a | 0 a a 0 0 0 a a | a | α |
| b | b β c 0 a γ 1 α | b | 0 b 0 b c b 0 a | b | β |
| c | c γ b a 0 β α 1 | c | 0 c a b c b a 0 | c | γ |
| α | α a 1 γ β 0 c b | α | 0 α 0 b c α γ β | α | a |
| β | β b γ 1 α c 0 a | β | 0 β a 0 0 γ β β | β | b |
| γ | γ c β α 1 b a 0 | γ | 0 γ 0 0 0 γ γ γ | γ | c |

This eight-element system P forms a non left-distributive quasi-Boolean ring of type R-D and it is easy to see that this system satisfies the postulates of Set II_2^* except the postulate B_1 .

If we apply Theorem 12 to the system M' whose addition and multiplication matrices are the transposed matrices of these of the four-element system M given above, then we obtain the eight-element system P' which satisfies the postulates of Set II_2^* except the postulate B'_1 . And this system P' satisfies also the postulates of

Set I_2^* except the postulate A_1 .

Next we shall construct a system \bar{P} (or \bar{P}') which satisfies the postulates of Set II_1 except the postulate B_1 (or B'_1). We have tried it in following the known method of converting the Boolean ring with unit into the Boolean algebra in replacing the operations $+$ and \times by the operations \vee and \wedge defined by $a\vee b = a + b + ab$ and $a\wedge b = a \times b$. But this method does not lead us to a complete success, we must modify it as follows:

We shall begin with constructing \bar{P} ; for the purpose we form a new algebraic system \bar{M} with same four elements as M with two binary operations \vee and \wedge in such a way that \wedge -operation in \bar{M} will be just the same of the multiplication in M , $a\wedge b = a \times b$; i.e. the \wedge -matrix of \bar{M} will coincide with the multiplication matrix of M . The \vee -matrix of \bar{M} will be symmetric and the lower part (the part under the principal diagonal inclusive of the diagonal itself) will be derived from the lower parts of the matrices of M by the rule $x\vee y = x + y + xy$. Thus we obtain M :

$$\begin{array}{c|cccc}
 \vee & 0 & a & b & c \\
 \hline
 0 & 0 & a & b & c \\
 a & a & a & c & c \\
 b & b & c & b & c \\
 c & c & c & c & c
 \end{array}
 \qquad
 \begin{array}{c|cccc}
 \wedge & 0 & a & b & c \\
 \hline
 0 & 0 & 0 & 0 & 0 \\
 a & 0 & a & 0 & 0 \\
 b & 0 & 0 & b & c \\
 c & 0 & a & b & c
 \end{array}$$

We notice that both the left and right distributive laws $(x+y)z = xz+yz$, $z(x+y) = zx+zy$ hold as far as $(x+y)z$, $xz+yz$, $z(x+y)$, $zx+zy$ can be found in the lower parts of the pair of matrices of M . We shall express this fact in saying that the pair of matrices of M have "regular" lower parts. Generally, we shall call the parts of a pair of matrices giving the results of two operations ($+$ and \times , or \vee and \wedge) in an algebraic system *regular*, if both the left and right distributive laws hold as far as the both sides of the distributive identities are calculable with the aid of these parts. We verify easily that the pair of matrices of \bar{M} have regular lower parts. \bar{M} satisfies the postulates F' , D , D_1 , B' , B'_1 but not B_1 , E' , C' , C'_1 of Set II_1 and we cannot define $0'$, a' , b' , c' , in \bar{M} so that E' , C' , C'_1 are not satisfied, only the operations \vee and \wedge of \bar{M} must be rewritten as $+$ and \times respectively in this case.

We observe further that the following parts (except the blank parts) of the matrices of P are regular. These parts are concerned with the elements of P which form a Boolean ring with unit according to the regular parts of M [3, 4: Sets S , S' , and II], and Stone's Theorem 1 [2].

| | |
|---|---|
| + | 0 1 a b c α β γ |
| 0 | 0 1 a b c α β γ |
| 1 | 1 0 α β γ a b c |
| a | a α 0 1 |
| b | b β c 0 γ 1 |
| c | c γ b α 0 β α 1 |
| α | α α 1 0 |
| β | β b γ 1 c 0 |
| γ | γ c β α 1 b a 0 |

| | |
|---|---|
| × | 0 1 a b c α β γ |
| 0 | 0 0 0 0 0 0 0 0 |
| 1 | 0 1 a b c α β γ |
| a | 0 a a 0 |
| b | 0 b 0 b 0 |
| c | 0 c a b c b a 0 |
| α | 0 α 0 α |
| β | 0 β a 0 γ β |
| γ | 0 γ 0 0 0 γ γ γ |

By the rules $x \vee y = x + y + xy$, $x \wedge y = x \times y$, we obtain from these the following regular parts of matrices.

| | |
|---|---|
| ∨ | 0 1 a b c α β γ |
| 0 | 0 1 a b c α β γ |
| 1 | 1 1 1 1 1 1 1 1 |
| a | a 1 a 1 |
| b | b 1 c b I α 1 II |
| c | c 1 c c c 1 1 1 |
| α | α 1 1 III α IV |
| β | β 1 β 1 III 1 β IV |
| γ | γ 1 β α 1 α β γ |

| | |
|---|---|
| ∧ | 0 1 a b c α β γ |
| 0 | 0 0 0 0 0 0 0 0 |
| 1 | 0 1 a b c α β γ |
| a | 0 a a 0 |
| b | 0 b 0 I b 0 II |
| c | 0 c a b c b a 0 |
| α | 0 α 0 III α IV |
| β | 0 β a 0 III γ β IV |
| γ | 0 γ 0 0 0 γ γ γ |

To fill out the blanks I, II, III, IV, we proceed as follows:

As to I, we shall put in simply the corresponding parts of the matrices of \bar{M} .

We now form two Hasse diagrams Figs. 1 and 2. Fig. 1 represents the Boolean algebra with eight elements 0, a, b, c, α, β, γ, and 1 which satisfies the relations \vee and \wedge between the regular parts of matrices in the above. The diagram of Fig. 2 is made as follows: the \wedge -relations among the elements 0, a, b, c are the same as in the upper part of \bar{M} , and the \wedge -relations among (0, γ, β, 1), (0, α, β, 1), and (0, b, α, 1) are respectively the same as in Fig. 1.

We fill out the blank IV according to Fig. 1. Then it is clear that the parts concerning to 1, α, β, γ are regular, as these elements form a sublattice of the Boolean algebra of Fig. 1.

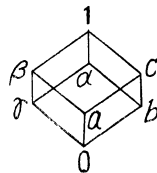


Fig. 1

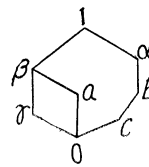


Fig. 2

Thus we obtain the following system \bar{P} filling up the blanks in this manner.

| | | | | | | | | | | | | | | | | | | | | |
|----------|----------|---|---------|----------|---|----------|---------|----------|----------|---|----------|---|---|---|----------|----------|----------|----------|----------|---|
| \vee | 0 | 1 | a | b | c | α | β | γ | \wedge | 0 | 1 | a | b | c | α | β | γ | a | a' | |
| 0 | 0 | 1 | a | b | c | α | β | γ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | a | b | c | α | β | γ | 1 | 0 | |
| a | a | 1 | a | c | c | 1 | β | β | a | 0 | a | a | 0 | 0 | 0 | a | 0 | a | α | |
| b | b | 1 | c | b | c | α | 1 | α | b | 0 | b | 0 | b | c | b | 0 | 0 | b | β | |
| c | c | 1 | c | c | c | 1 | 1 | 1 | c | 0 | c | a | b | c | b | a | 0 | c | γ | |
| α | α | 1 | 1 | α | 1 | α | 1 | α | α | 0 | α | 0 | b | c | α | γ | γ | α | a | |
| β | β | 1 | β | 1 | 1 | 1 | β | β | β | 0 | β | a | 0 | 0 | γ | β | γ | β | b | |
| γ | γ | 1 | β | α | 1 | α | β | γ | γ | 0 | γ | 0 | 0 | 0 | γ | γ | γ | γ | c | |

It is easily verified that \bar{P} satisfies the postulates of Set II₁ except the postulate B₁, only the \vee, \wedge operations of \bar{P} must be rewritten as $+$ and \times respectively in this case.

The transposed matrices of these matrices of \bar{P} define a system \bar{P}' satisfying the postulates of Set II₁ except the postulate B'₁.

3. Independence proofs of the four new sets for Boolean algebras.

The independence of the postulates of Set I₁, Set II₁, Set I₂^{*}, and Set II₂^{*} will be established by the following K-systems, each of which satisfies all postulates except the one indicated by the number and the postulate of the system; for example K_{I₁}F' is the independence-system of postulate F' in K of Set I₁.

For the independence proofs of Set I₁ and Set II₁ we can use the examples of two-element system which were given by Bernstein [7: p. 160], only the operation \vee must be rewritten as $+$ in the present cases.

- K_{I₁}F', K_{II₁}F' : IF⁷⁾
- K_{I₁}D, K_{II₁}D : ID
- K_{I₁}D₁, K_{II₁}D₁ : ID₁
- K_{I₁}E', K_{II₁}E' : IE
- K_{I₁}A : IA
- K_{I₁}A₁ : IA₁
- K_{I₁}B', K_{II₁}B' : IB
- K_{I₁}B₁ : IB₁
- K_{I₁}C, K_{II₁}C' : IC
- K_{I₁}C₁, K_{II₁}C'₁ : IC₁
- K_{II₁}B₁, K_{II₁}B'₁ : The examples were given in the preceding paragraph. Here $0 = a(a+c) \neq aa+ac = a$, and $0 = (a+c)a \neq aa+ca = a$ respectively.

| | | | | | | | | | | | | | | | | | | | | |
|---|----|--|---|---|---|---|---|---|---|---|---|---|---|---|---|----|---|---|---|---|
| K _{I₂} [*] F', K _{II₂} [*] F' | : | IF | | | | | | | | | | | | | | | | | | |
| K _{I₂} [*] D, K _{II₂} [*] D | : | $+$ <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>0</td><td>1</td></tr> <tr><td>0</td><td>—</td></tr> <tr><td>1</td><td>0</td></tr> </table> \times <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>0</td><td>1</td></tr> <tr><td>0</td><td>0</td></tr> <tr><td>1</td><td>0</td></tr> </table> <table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td>a</td><td>a'</td></tr> <tr><td>0</td><td>1</td></tr> <tr><td>1</td><td>0</td></tr> </table> | 0 | 1 | 0 | — | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | a | a' | 0 | 1 | 1 | 0 |
| 0 | 1 | | | | | | | | | | | | | | | | | | | |
| 0 | — | | | | | | | | | | | | | | | | | | | |
| 1 | 0 | | | | | | | | | | | | | | | | | | | |
| 0 | 1 | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | | | | | | | | | | | | | | | | | | | |
| 1 | 0 | | | | | | | | | | | | | | | | | | | |
| a | a' | | | | | | | | | | | | | | | | | | | |
| 0 | 1 | | | | | | | | | | | | | | | | | | | |
| 1 | 0 | | | | | | | | | | | | | | | | | | | |

$$\begin{array}{l}
 K_{I_2}^* D_1, \quad K_{II_2}^* D_1 \quad : \quad + \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & - \\ \hline 1 & 0 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{1} \\
 \\
 K_{I_2}^* E', \quad K_{II_2}^* E' \quad : \quad + \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{-} \\
 \\
 K_{I_2}^* A \quad : \quad \text{IA}
 \end{array}$$

$K_{I_2}^* A_1, K_{II_2}^* B_1, K_{III_2}^* B'_1$: The examples were given in the preceding paragraph. Here $a=ac \ni ca=0, 0=a(a+c) \ni aa+ac=a$, and $0=(a+c)a \ni aa+ca=a$ respectively.

$$\begin{array}{l}
 K_{I_2}^* B_1 \quad : \quad + \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{0} \\
 \\
 \frac{a}{1} \quad \frac{a'}{1}
 \end{array}$$

Here $0=1(1+0) \ni 1 \cdot 1 + 1 \cdot 0 = 1$.

$$\begin{array}{l}
 K_{I_2}^* C, \quad K_{II_2}^* C' \quad : \quad + \quad \begin{array}{|c|c|c|c|} \hline 0 & 1 & a & b \\ \hline 0 & 0 & 1 & a & b \\ \hline 1 & 1 & 0 & b & a \\ \hline a & a & b & 0 & 1 \\ \hline b & b & a & 1 & 0 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|c|c|} \hline 0 & 1 & a & b \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & a & b \\ \hline a & 0 & a & 1 & b \\ \hline b & 0 & b & b & 0 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{1} \\
 \\
 \frac{a}{1} \quad \frac{a'}{0} \\
 \frac{a}{a} \quad \frac{a'}{b} \\
 \frac{a}{b} \quad \frac{a'}{a}
 \end{array}$$

Here $b=0+ab=ab+0 \ni 0$.

$$\begin{array}{l}
 K_{I_2}^* C_1, \quad K_{II_2}^* C'_1 \quad : \quad + \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{0} \\
 \\
 \frac{a}{1} \quad \frac{a'}{1}
 \end{array}$$

Here $0=1(0+0)=(0+0)1 \ni 1$.

$$\begin{array}{l}
 K_{I_2}^* G, \quad K_{II_2}^* G \quad : \quad + \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \quad \frac{a}{0} \quad \frac{a'}{1} \\
 \\
 \frac{a}{1} \quad \frac{a'}{0}
 \end{array}$$

Here $1=(0+1)+1 \ni 0$.

$K_{I_2}^* H, K_{II_2}^* H$: The example was given in the preceding paragraph. Here $a=a(ab) \ni (aa)b=1$.

References

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